Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 58
Linear first-order ODEs

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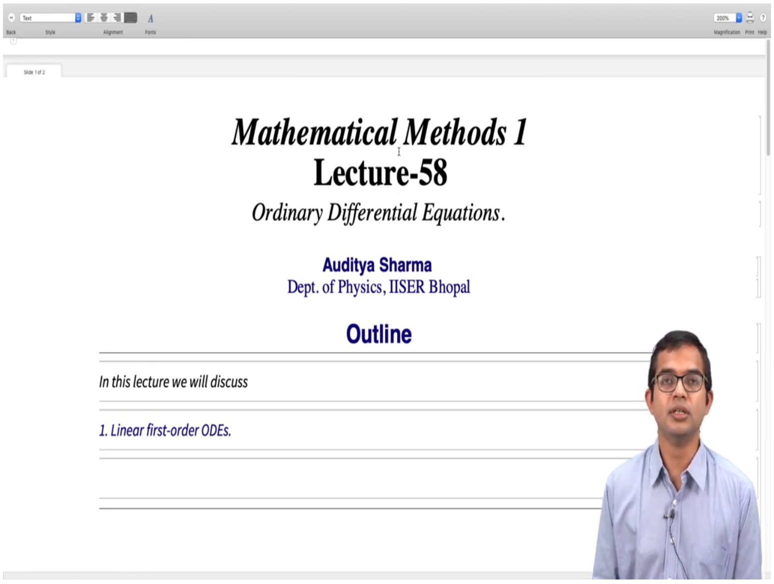

So we have seen how first order differential equations when they are separable can be solved formally; you know by collecting terms in a nice way and just integrating. Now, in this lecture we will see how if a first order ODE happens to be linear, there is a nice prescription that we can give for a general problem of this kind. Linear first order differential equations can be solved exactly as will be described in this lecture ok.

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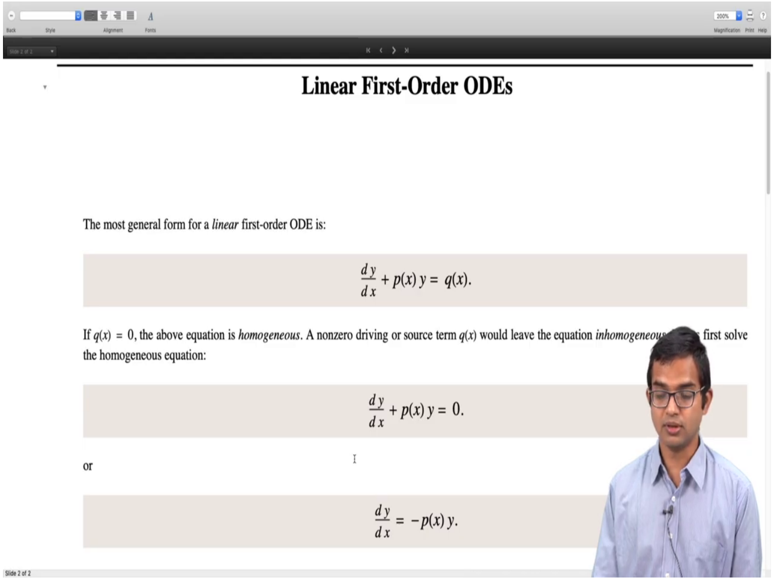

So, the most general form for a linear first order ODE is d y by d x plus some function of x, we will call it p; p of x times y is equal to another function of x which is q of x right; so this is as general as it gets. So, we have seen that if q of x is equal to 0, then this differential equation is also a homogeneous differential equation. And on the other hand, if q of x is present which is also sometimes called a driving term right, then it becomes an inhomogeneous differential equation.

So, in general; so when you are interested in solving an inhomogeneous differential equation, we should first solve the corresponding homogeneous equation and then see what happens, if you also have a driving term. So, in the spirit of this philosophy, so let us first look at the solution for the problem d y by d x plus p of x, y is equal to 0. So, this turns out to be a separable differential equation; so you can bring it to this form d y by d x is equal to minus p of x times y.

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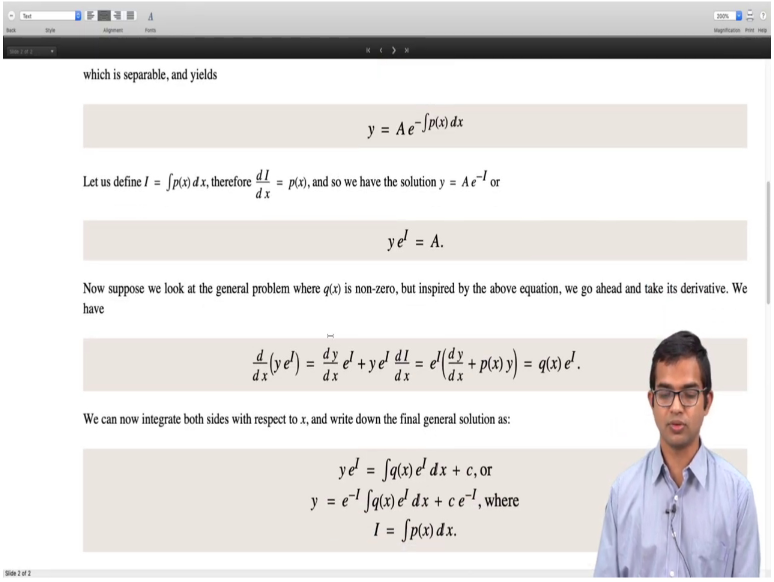

And then you bring y to the left hand side and then d x to the right hand side and then you have a formal solution; y is equal to A times e to the minus integral p of x d x right, straight forward.

Now, we will look at this quantity called I, we will define I is equal to integral p of x d x. It turns out that this quantity will be useful even when you have the you know more general problem which involves a forcing term right. If the forcing term is nonzero, you can still you know consider this quantity I. Let us define I is equal to integral p of x d x; therefore, the derivative of this quantity is p of x right. So, this is just by definition if the integral of p of x d x is I d I by d x must be equal to p of x.

So, we have the solution for this problem is y is equal to A times e to the minus I right. So, or in other words; y times e to the I is a constant right if the right hand side did not have a forcing term. But let us see; let us look at this quantity y times e to the I and then take the derivative. So, think of this quantity right, it is a completely legitimate quantitative study and let us see what its dependence is; if you plug back in the forcing term right.

So, now suppose q of x is nonzero, but you can still consider y times e to the I and take its derivative with respect to x. So, if we take the derivative of this quantity with respect to x right. So, we have this you know rule for differentiation of the product of two functions. So, we have d y by d x times e to the I which is you know treated like a constant plus y times e to the y times the derivative of e to the I which is nothing, but e to the I times d I by d x.

Now, because it is just exponential; you can pull this out, you know you get the same factor or in both the terms, you pull this out and then you have d y by d x plus p of x times y inside brackets right. But, if you look at this whole quantity carefully, this we will see is actually nothing but this quantity; the left hand side of the original differential equation right.

So, if so this is nothing but the left hand side of the original differential equation right which would be 0, if the forcing term is 0. And indeed it is not a surprise because if y times e to the I is equal to; is equal to a constant, the derivative has to be 0. But here in the more general case; it is not going to be 0, but we will replace the stuff inside this bracket with just q of x. So, we have d by d x of y times e to the I is equal to q of x times e to the I.

So, now what we have is on the left hand side, we have an exact derivative of some function and on the right hand side, we have a function of x alone. So, we can integrate both sides, the left hand side is easy to integrate because we have already given the derivative of some function. So, y times e to the I is equal to the integral of a function which is purely a function of x on the right hand side; so you get the integral of q of x e to the I, d x plus some constant right.

And so we are left with; now we can bring back this e to the I to the; you know multiply throughout with e to the minus I. So, then you are just left with y is equal to e to the minus I integral q of x e to the I dx plus c times e to the minus I where I is also given you know which we define as integral p of x d x.

So in fact, this box is the prescription; this is the general solution for a first order linear differential equation ordinary differential equation right. So, you just find this I you know multiply throughout with e to the I and then you know you see the you know identify this left hand side to be an exact derivative and then just integrate and you are done right.

So, let us look at how this operates in practice. So, let us solve this differential equation 1 plus x squared d y by d x plus 4 x y is equal to x; so I am given this differential equation.

So, the first step is to bring it to this canonical form right, we have worked out this theory in a particular form. So, we have to bring this differential equation into that form.

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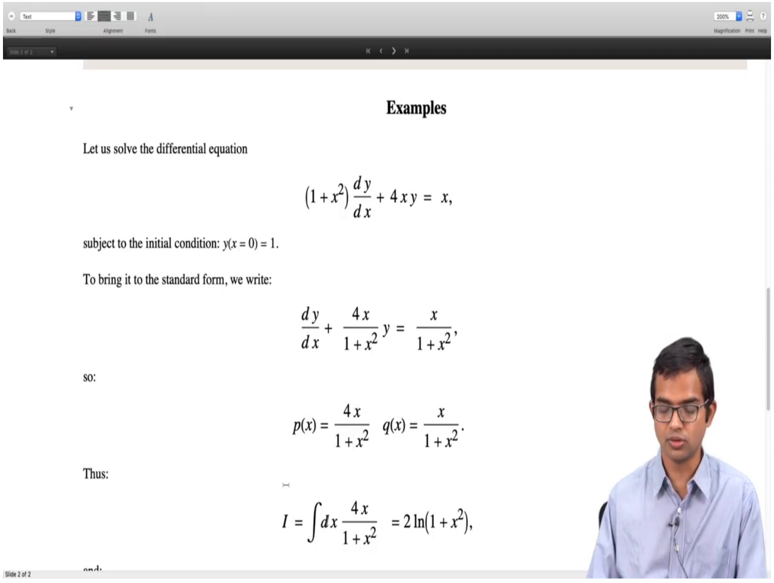

So, in order to do that ok; so we were also given this initial condition. So, there is always a free constant associated with a first order linear differential equation and that free constant is fixed if you are given; if you are given also an initial condition right, y of x equal to 0 equal to 1 is given in this case ok.

So, to bring it to the standard form we just divide throughout by 1 over; divide throughout by 1 plus x squared. So, we have d y by d x plus 4 x divided by 1 plus x squared y is equal to x over 1 plus x squared right. So, p of x is equal to 4 x divided by 1; 1 plus x squared. So, immediately we identify that this is connected to the form that the general form that we worked with and so p of x here corresponds to 4 x divided by 1 plus x squared and q of x here corresponds to x over 1 plus x squared right.

So, although we could just directly sort of use this prescription as it is and you know keep all this stuff in our head, it is more transparent to just you know compute I and multiply throughout by e to the I. So, I is equal to integral d x 4 x divided by 1 plus x squared and this we immediately see is nothing but you know log of 1 plus x squared times 2, if I take a derivative of this; I get 1 over 1 plus x squared, then 2 x; multiplied by 2 will give me 4 x. So, this is you know the integral of this right.

So, you might ask: there should be a free constant available, but that is not a problem because this is just the sort of intermediate step and we will anyway get this free constant at the end. So, as far as this you know when you are finding I you can choose whatever constant you want and you can take it to be 0 right when there is no loss of generality.

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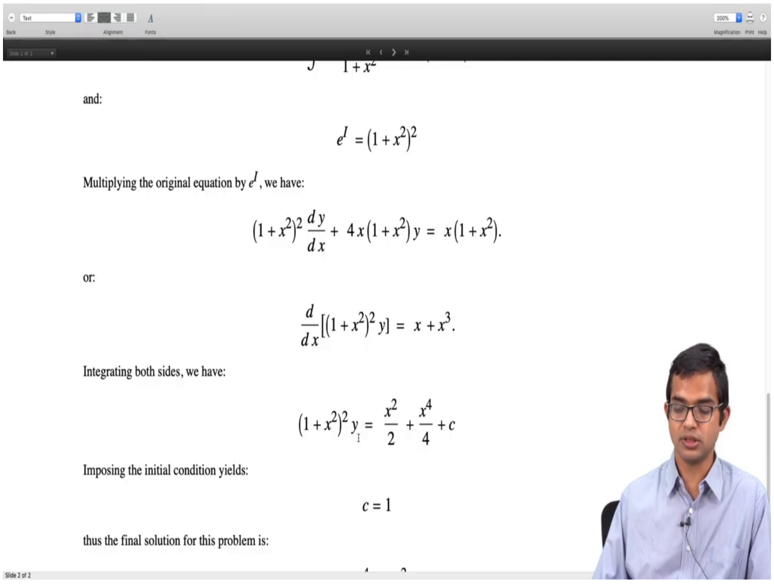

And e to the I is 1 plus x squared; the whole squared right. So, this follows directly from you know this what I is multiplying throughout with e to the I, we have 1 plus x squared, the whole squared times d y by d x plus one of these 1 plus x squared, you know in the denominator will cancel.

So, then you have plus 4 x times; 1 plus x squared; times y is equal to x times 1 plus x squared right. So, now, a careful look at this differential equation in this form; immediately tells us that in fact, the left hand side can be written as an exact derivative right. So, this is not a surprise because that is the whole point of this e to the I you are multiplying throughout to bring it in this special form.

So, d y by d x of 1 plus x squared whole squared times y; you can check this, if you take a derivative; you will get this you know 2 into 2 4 x times 1 plus x squared times y, that is the second term. And the first term is you know; you treat 1 plus x squared the whole squared as it is and take the derivative with respect to the d y by d x.

And on the right hand side it is convenient to expand it out and write it as x plus x cube integrating both sides, we have you know the left hand side is super easy, it is just 1 plus x squared whole squared times y. Right hand side will give you x squared by 2 plus x to the 4 by 4 plus this constant right.

So, you do get this free constant in the end in any case. So, that is the reason why when you are finding I you can afford to be you know sloppy about not including any see and it is not even being sloppy because in the end we anyway taking care of this there is only one free constant you will get in the final answer.

So, the initial condition in fact, removes that free constant as well. So, in this case we get the free constant the initial condition will force c to be 1 because you have to have when you put x equal to 0 you know the left hand side y must be equal to 1, so c is equal to 1.

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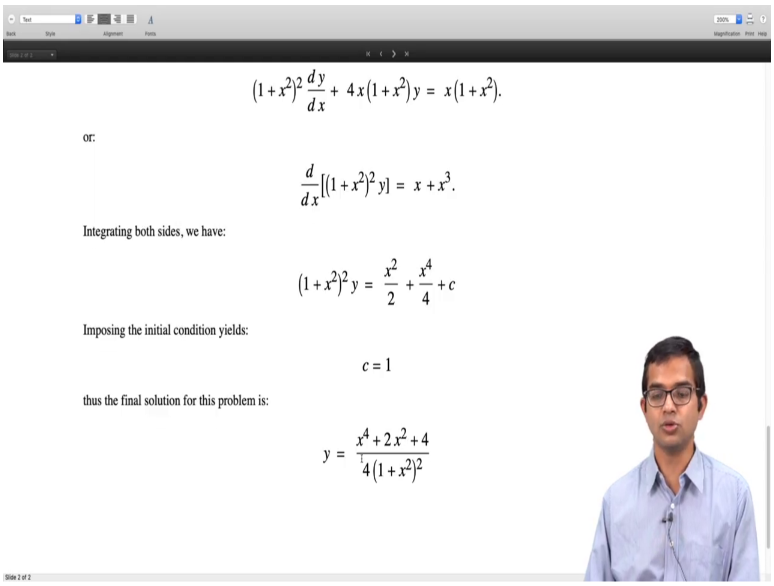

So, if you plug this in you see that the final complete solution for this problem with the given initial condition is x to the 4 plus 2 x squared plus 4, divided by 4 into 1 plus x squared the whole squared right.

So, already we have a fairly powerful tool, so you can play with you know differential equations of this kind, you can come up with your own variants of this you know come up with p of x's and and q of x's and see how you know this whole method plays out right.

So, this is like a prescription which is solidified by working out many example problems right. So, in this lecture we have discussed how to solve an ordinary differential equation which is linear and of first order; that is all for this lecture.

Thank you.