Mathematical Methods 1

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Ordinary Differential Equations
Lecture - 56
Ordinary Differential Equation

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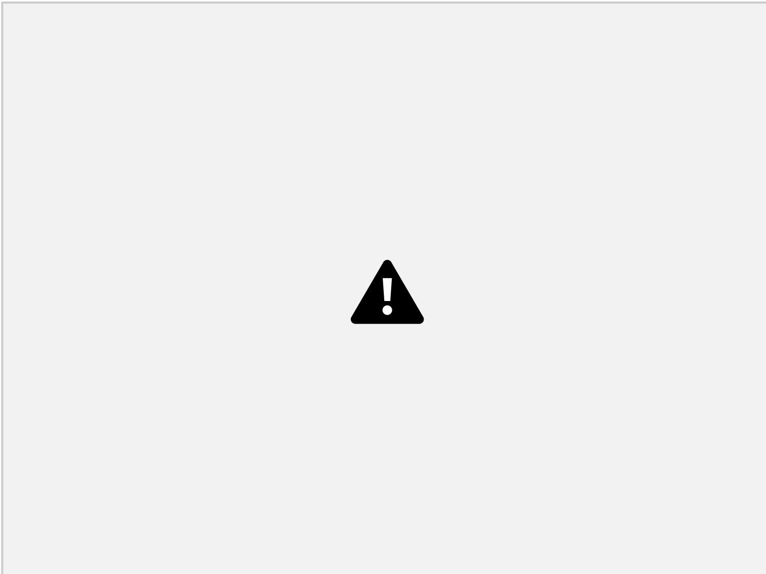

Ok starting from this lecture we move on to the next topic which is Ordinary Differential Equations. So, in this lecture, we will you know discuss the nomenclature ordinary, why is, what is ordinary about ordinary differential equations. We will also look at some basic classification of differential equations. And then we will set up the scene for our study of differential equations ok.

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So, a differential equation you know contains in general many derivatives and they are tied together in some you know some linear combination of these derivatives of some unknown function, and then they add up to some other function or it can add up to 0, right. And our job is to find a function which satisfies such a you know linear combination of various derivatives. And it is an ordinary differential equation if all the derivatives involved are ordinary, meaning there are no partial derivatives, right.

So, if functions depend on more than one variable, right and you know their derivatives with respect to different independent variables are also contained in your differential equation that is the more complicated kind and that is a partial differential equation. So, you know in general in physics we have lots of real world phenomena which actually give us partial differential equations.

And then it's possible, right, in using certain techniques for certain class of partial differential equations to be able to recast you know these partial differential equations into ordinary differential equations or you know a smaller set of partial differential equations can be you know recast as a larger set of ordinary differential equations.

In any case, the study of ordinary differential equations and a solid theoretical understanding of ODEs is absolutely essential if we want to tackle the more gory problems which come up within the field of partial differential equations ah. So, and the theory of ODEs is an extremely well developed, very beautiful theory, and so, our job in the next several lectures is to make a thorough study of ODEs, right from the point of view of a physicist, but we will also go into the theatrical details, right.

So, ok so let us start with you know classification of ODEs, right. So, one immediate and very important class of ODEs that we will concern ourselves with is the class of linear ODEs, right. So, linear ordinary differential equations are those in which the unknown function appears only in a linear fashion, right, the unknown function and its derivatives. All of them are to be linear. Such differential equations are called linear differential equations. So, let us look at a whole bunch of examples.

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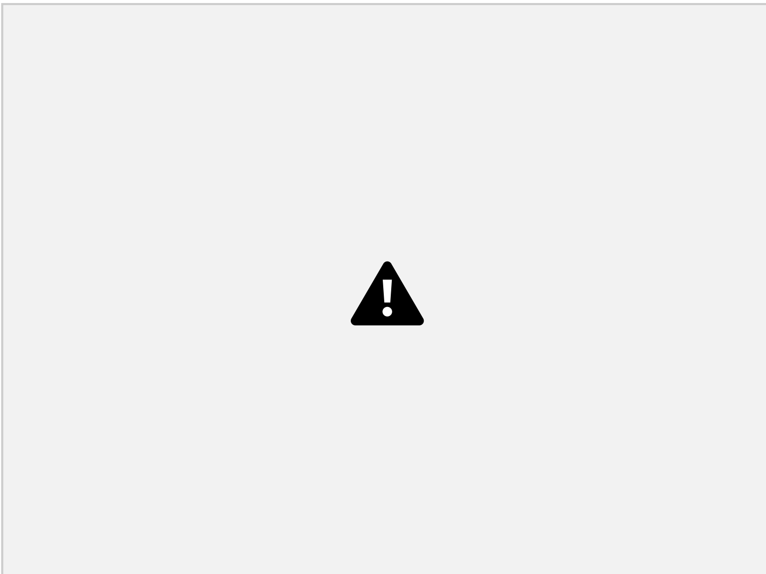

So, I have all of these equations which I am claiming are linear equations, right. So, we have d squared, we have the second order derivative of y appears here, but there is no square of this quantity. So, it is linear in the second order derivative, first order derivative, first and in the function itself all are linear therefore, this is the linear differential equation.

Now, you can have x here, you can have you know the independent variable can be you know very crazy non-linear is allowed. So, you know an example of non-linearity is here a very simple non-linearity you have x, non-linearity in the independent variable. So, and that is allowed. And that is not really; so, you would not say that this differential equation is nonlinear - it is a linear differential equation, right.

So, you know when you are starting out this is a point of confusion for a lot of students, you know, they confuse between the x's and y's, the dependent variable and the independent variable you should look at it carefully. And indeed, this differential equation is also linear because although we have x squared and in fact, you know much more violent something like sin of x appears it does not matter as long because y and dy by dx both of these are linear.

So, likewise this also is a; so, x squared can appear here x can appear here, sin of x can appear here, cos x can appear here all kinds of possibilities as long as it's only to do with x. So, all of these are linear differential equations. So, let us look at a few examples of nonlinear differential equations to contrast against the first set. So, now, you see that if you have a sin of y, then that is trouble. So, that is a nonlinear differential equation.

So, if you have y square that is also a problem. And dy by dx the whole squared plus x y is equal to 1 is also a nonlinear differential equation. So, in general, nonlinear differential equations are very hard problems and you know there are very small very special subclasses of problems only amenable to exact solutions.

And even when there are solutions there is so much complexity, so in fact, the field of nonlinear differential equations is a whole subject by itself, right. There are so many rich phenomena associated with non-linear differential equations that it is a whole topic in itself. Lots of exciting phenomena come out of this, right.

It is not the focus here for us to go into all of those properties, but our goal here will be to make a generic study of differential equations, right. So, this is one classification at this point.

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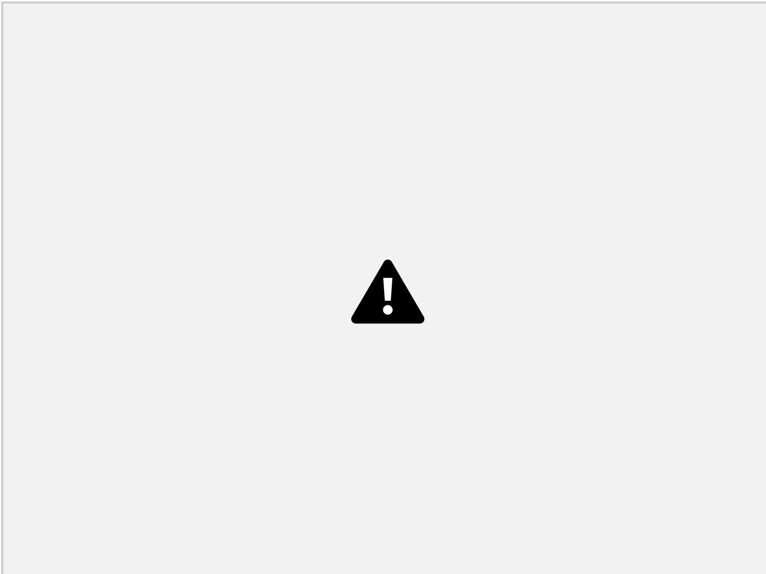

So, another; so, ok. So, linear ODEs have this very special form. You know there is a linear operator it acts upon you know the function, the unknown function y of x and it can give you some other function of x. So, that is allowed, right. So, this operator is linear. And so, a key property of linear differential equations is the so-called superposition principle.

So, but ok before we come to that let us also quickly point out you know another, I guess a definition or you know further classification of linear ODEs, is you know those linear ODEs in which this right-hand side f of x is equal to 0 they are called homogeneous, if there is no or there is no driving term, right.

So, inhomogeneous differential equations are also going to be studied in our theory. And oftentimes there is a close connection between the solution of the corresponding homogeneous differential equation and the full blown inhomogeneous differential equation, right. So, in that context, it is useful to be able to first study the simpler problem where you put f of x to be 0, and then you bring in f of x and then you work out the full more general solution to the more general differential equation, right.

So, in general, let us consider an explicit form for a general linear ODE, right. You know you can write it like this: a naught y plus a 1 y prime plus a 2 y double prime, so on all the way it can go to whatever order equal to b provided the a's and b's are either constants or functions of x alone, right. So, that is important, right. You can have functions of x and that is not a problem. It still will be called a linear differential equation.

So, when you have a linear ODE the superposition principle holds which means that if you can find two different solutions then an arbitrary linear combination of these two solutions is also a solution of the ordinary differential equation, right. So, this will break down, if there is any non-linearity in y, right.

You can take up some simple example, and check it for yourself, so the thing about linear differential equations is that superposition means of solutions are also solutions, right. So, it is a key aspect of linear ODEs.

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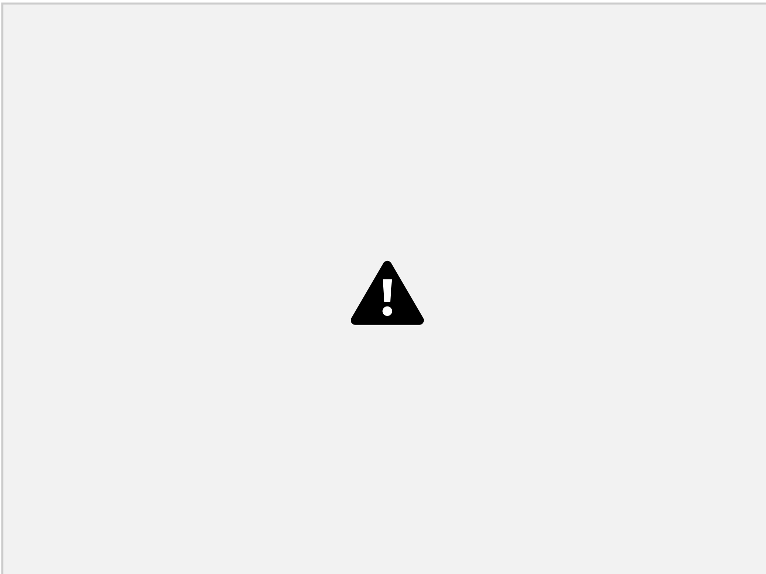

Now, there is another notion which we have sort of already mentioned, but let us make that explicit and that is the order of ODE, right. So, sometimes you know particularly early on in the study of differential equations students get confused between order and linearity or non-linearity, right. So, let us look at a few examples. So, the order of a differential equation is the order of the highest derivative of the unknown function, right.

So, let us take another look at the examples that we have already looked at, right. So, we took all these examples and broke them into two bins, one of which contains linear equations and the other bin contains nonlinear differential equations, but we could have also broken this down into you know order 1, order 2, and so on. So, the examples of differential equations with order 1, right from our collection of differential equations are all of these.

So, if x square dy by dx plus x squared y equal to sin x, you know there is no problem if you have sin of y, it is all good as long as you know you just look for the highest derivative. So, as far as this classification is concerned you only look at the highest derivative and you see dy by dx the whole squared appears it is not a problem, right.

So, there is no second order. It is the first order differential equation. It is a nonlinear differential equation, but it is a first order differential equation. And so, likewise you have dy by dx and you have y squared, right. So, it is a nonlinear differential equation, but it is first order.

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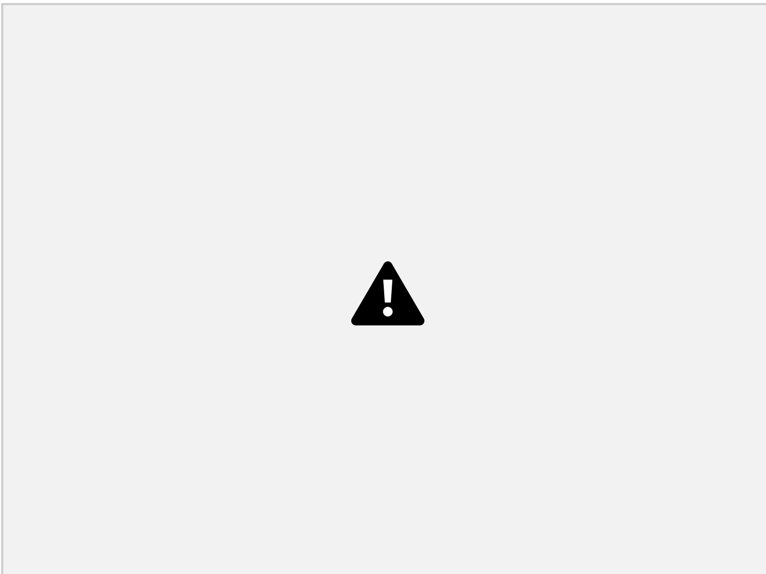

And order 2, right. So, this is order 2 and linear and this is also order 2 and linear, order 2 and once again it is linear, right. So, you can cook up examples where you know you can put in a sin of y or you know sin squared of y cube, I do not know. So, you can come up with all kinds of very complicated terms. I mean you may not be able to solve them, right. At this point it is only an exercise in classifying differential equations. So, you can construct as complicated a differential equation as you want and identify their order, ok.

Now, order 5 I have just you know made up one other example. So, I can think of something like x to the 5, d to the 5 y by dx to the 5 plus x to the 4 and so on, all the way up to y is equal to cos of x, right. So, I could put whatever function of x on the right-hand side, it is still an order 5. So, as long as there is this d to the 5 by dx 5; even if all of these other terms are absent it is still 5th order.

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And so, in general, so you can say that a linear differential equation of order n has the following general form. So, now, I am saying I am putting a linear differential equation. So, I have a n of x d n y by dx n plus a n minus 1 of x d n n minus 1 y divided by dx to the n minus 1, so on all the way up to a naught of x y is equal to b of x.

So, the essential requirement for this to be order n is that a n of x should not be identically 0, if it were then it would be a differential equation of a lower order, right. So, a n of x is definitely not 0, right. It could be a constant. It would be some other non-trivial function of x does not matter, right.

And all of these other coefficients are also you know you are free to choose them to be functions of x, no matter how complicated, no matter how non-linear in x, it is not a problem, but it is still this overall differential equation would be linear, and it is order n, if a n of x is nonzero ok.

So, that is an introduction to ODEs, and why we call them ODEs and some basic concepts pertaining to order, and so on. So, that is all for this lecture.

Thank you.