Mathematical Methods 1
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Fourier Transforms
Lecture – 55
Parseval's theorem for Fourier series

Ok. So, in this lecture we will look at a generalization of Parseval's Theorem for Fourier Integrals. So, we saw how you know if you are able to write a function as a Fourier series, then you know that the average of the square of the function right is related to the sum of the squares of all these coefficients. Or if you are writing it in the exponential series, then it is the sum of the square of the moduli of these the coefficients, right.

So, there is an extension of you know this theorem which is applicable also for Fourier transforms and that is going to be the content of this lecture. And, once again just like with Parseval's theorem with Fourier series, this too has some very nice applications which follow from Parseval's theorem, right. So, that is what is coming up in this lecture.

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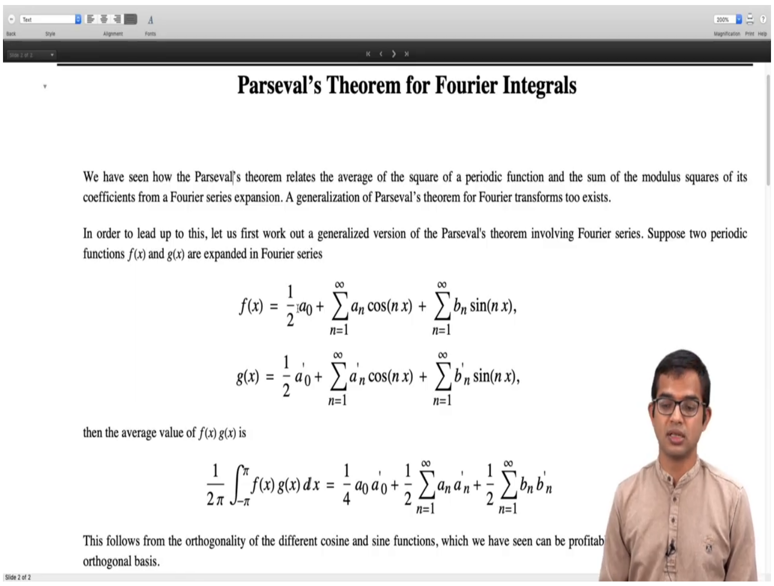

So, in order to you know write down Parseval's theorem for Fourier integrals you know first we will work out a generalized version of Parseval's theorem involving Fourier series for a periodic function itself, right. So, we saw how you know the product of a function with itself and if you averaged over the of this quantity was given in terms of the coefficient.

So, in fact, what you could do is consider two periodic functions f of x and g of x expand each of them which are both periodic with the same period let say, 2 pi in this case and f of x can be expanded in a Fourier series g of x is expanded in another Fourier series. So, you have all these coefficients a n and b n for the first function and a n prime and b n prime for the second function.

So, then instead of considering the average of f of x squared or the average of g a of x squared you can also consider the average of f of x times g of x. So, if you do this right so, you can show that in fact, this average value 1 over 2 pi integral minus pi to pi f of x g of x dx is given by one over 4 a naught a naught prime, right. So, only the diagonal terms so to speak will remain, right.

So, a n will go with a n n prime and then you get this customary factor of half and likewise b n will go with b n prime of an overall factor of half outside and there is a summation involved over all n both for the these a n terms and for the b n terms, right. It is not hard to you know show this result, right and I urge you to convince yourself that this works out.

So, the key idea is of course, that you know these functions are orthogonal to each other right. So, they are members of this basis which is an orthogonal basis we have seen this and so, if you take any cosine n x and multiply it with some other cosine of m x as long as n and m are not the same you know the average value of this product is going to go to 0.

But, on the other hand if it is if n and m are the same then you get you know just one in the well, you get a half right. So, it is the average value of cos squared in a period. So, you will get a half. So, these are orthogonal, but not normalized. So, it is straightforward to work out this integral of any of these basis elements with itself. The inner product of any element with itself is straightforward to work out.

And, likewise also with sin squared of n x you are going to get half, but sin n x times sin m x you know the average value is 0, right. So, this is an immediate consequence of the same result we use to prove Parseval’s theorem you know involving just a single function.

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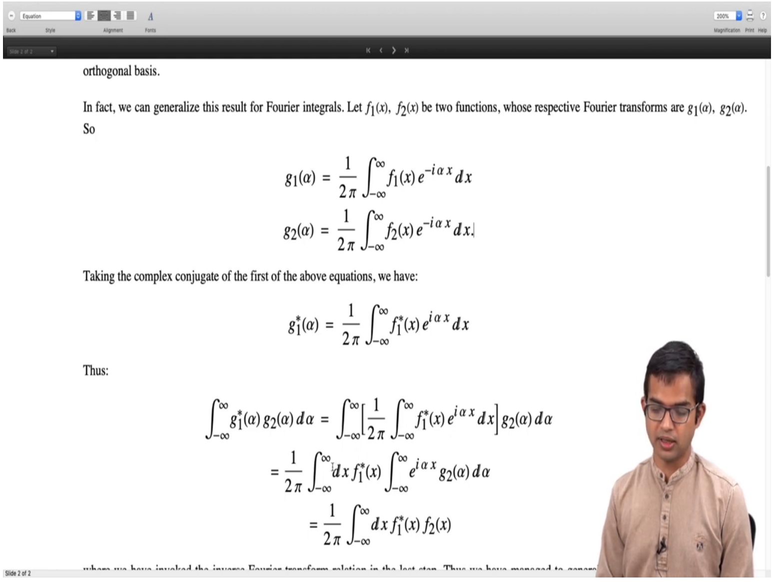

Now, it turns out this result in fact, generalizes to Fourier integrals, right. So, let say you have f of f 1 of x and f 2 of x are two big non-periodic functions and you find their Fourier transforms and let say they are g 1 of alpha and g 2 of alpha. So, we have g 1 of alpha is 1 over 2 pi integral minus infinity to plus infinity f 1 of x, e to the minus i alpha x dx and likewise you have another expression for g 2 of alpha, right.

So, if you take the complex conjugate of the first of the above equations, right so, the generalization of you know this result for Fourier series you know one has to be a little more careful and introduce has to work with complex conjugates of one of these functions, right.

So, like we discussed when we were doing linear algebra, if you have a real field then so, inner product of these kinds of vectors you can just think of them as the integral minus pi to pi f of x g of x dx, for example. But, if you allow for these functions to be complex then there is a complex conjugation associated with going to the bra-vector.

So, the inner product is going to be not just you know f of x times g of x or I mean in terms of alpha. So, you have to take the complex conjugate of one of these functions, right. So, f 1 star of x and f 2 of x will come in and f g 1 star of alpha and g 2 of alpha will come in as we will work out in detail.

So, let say you start with these definitions you have g 1 of alpha and g 2 of alpha. Now, you take the complex conjugate of the first of these equations. So, you have g 1 star of alpha is equal to 1 over 2 pi minus infinity to plus infinity f 1 star of x and then you have e to the i alpha x instead of e to the minus i alpha x dx of course.

Now, we multiply this g one star of alpha with g 2 of alpha and then integrate from minus infinity to plus infinity. So, g 1 star of alpha we already have this expression for g 1 star of alpha. So, we plug this in and then g 2 of alpha remains as it is, d alpha as it is integral of course, minus infinity to plus infinity.

So, then we see that 1 over 2 pi you know this f 1 star of x you know comes out. So, d x also comes out. So, you have d x f 1 star of x integral minus infinity to plus infinity e to the i alpha x g 2 of alpha d alpha. So, but so, I have just you know rearranged these you know various factors, right. So, this part does not care about you know this integral involving alpha. So, I am allowed to bring this stuff outside.

And, then now, I observe that in fact, this integral is nothing but the inverse Fourier transform of g 2 of alpha, but the inverse Fourier transform of g 2 of alpha is just f 2 of x. So, this is equal to 1 over 2 pi integral minus infinity to plus infinity dx f 1 star of x f 2 of x.

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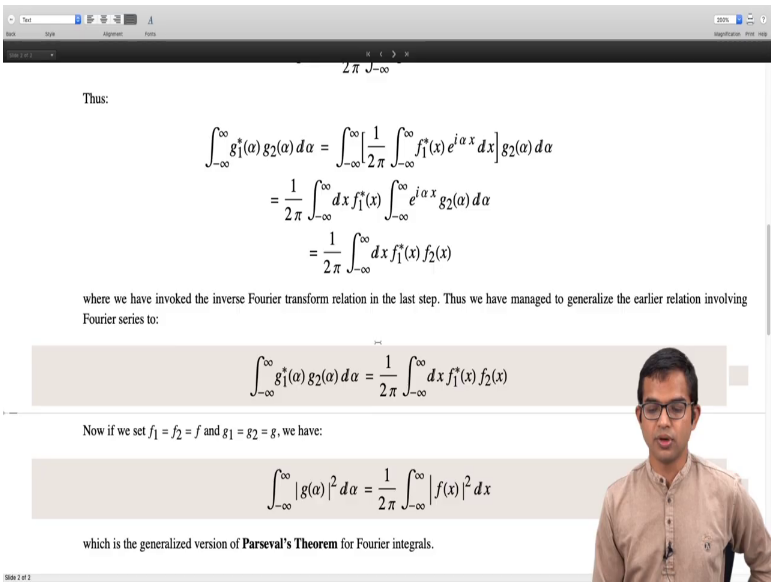

So, what we have managed to show is really an extension of the result for the Fourier series right involving two functions which were periodic, but now if you have two functions f 1 and f 2 which are not periodic. So, if you do this integral minus infinity to plus infinity dx f 1 star of x times f 2 of x and then put in this factor 1 over 2 pi this is going to be the same as the integral minus infinity to plus infinity g 1 star of alpha g 2 of alpha d alpha.

So, now if you say that you know these two functions are the same let say f 1 and f 2 are the same and they are both equal to f, then of course, the Fourier transforms also are going to be the same. g 1 equal to g 2 equal to g then, we have integral minus infinity to plus infinity mod of g of alpha squared d alpha is equal to 1 over 2 pi integral minus infinity to plus infinity mod of f of x square dx so, which is basically the Parseval's theorem. So, this is the generalization of Parseval's theorem for Fourier integrals, right.

So, this factor you know is of course, dependent on the manner in which you have defined your Fourier transform and inverse Fourier transform, but like I said before you have the freedom to define your Fourier transform and then automatically it fixes the inverse Fourier transform for you, right and it is does not matter how you know you share this factor between the Fourier inverse Fourier transform as long as you are consistent throughout in your calculation ok.

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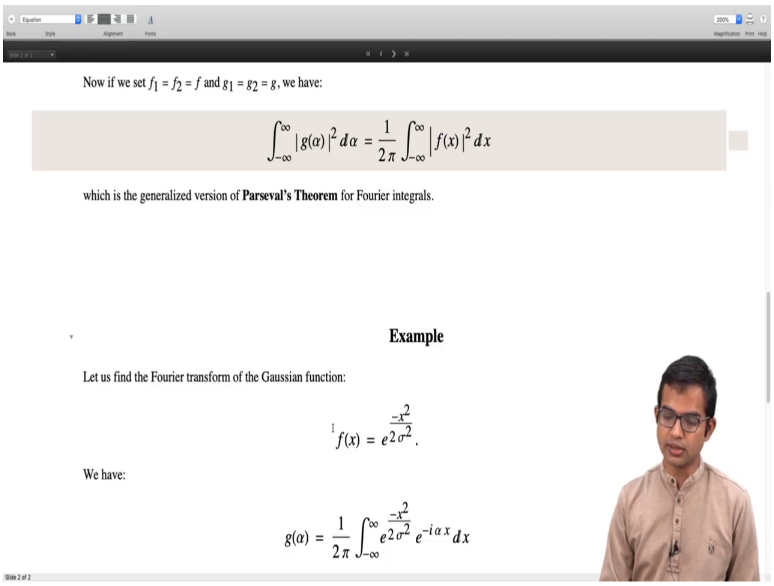

So, let us look at an example where we apply this idea and so, we are going to work out the Fourier transform of the Gaussian function, right. So, I have f of x is equal to exponential of e to the exponential of minus x squared by 2 sigma squared, right. So, well, I mean you have to read this carefully and understand that really I am referring to the exponential of minus x squared by 2 sigmas 2 sigma squared.

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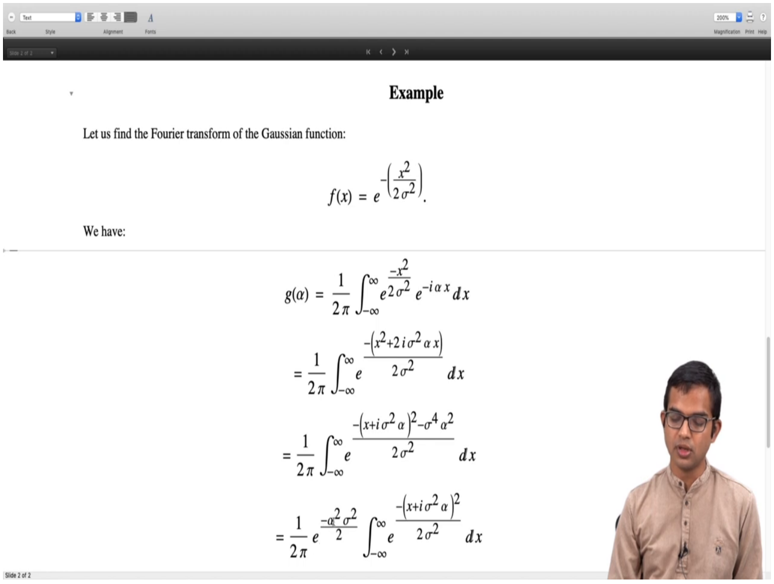

So, if I do this maybe it will look better. Yeah I guess this is a bit better, but so, the key point is that. So, this is what I am referring to, it is an exponential over all of this stuff and likewise. So, this is going to appear later on as well. So, if I want to find the Fourier transform of this function. So, this is the Gaussian function right. So, you have seen what it looks like. It is peaked at the origin and then it tapers off and it falls off you know very quickly on both sides and it is symmetric about the origin.

So, if you take the Fourier transform of this Gaussian function you have 1 over 2 pi integral minus infinity to plus infinity this function times e to the minus i alpha x dx. And, so, the way to proceed to evaluate this integral is to do what is called the completion of squares. So, you have an exponential of minus x squared plus 2i sigma squared alpha x divided by 2 alpha squared.

So, this you can rewrite this as x plus i sigma squared alpha the whole squared and then you have an extra constant part which has to be subtracted out and so, if you take you know if you carry out this exercise carefully.

So, you will get an extra factor 1 over 2 pi you know is already there then you have an exponential of minus alpha squared sigma squared by 2, and then you have this integral which is actually just a standard Gaussian integral except that you know x seems to be shifted by some constant complex number right.

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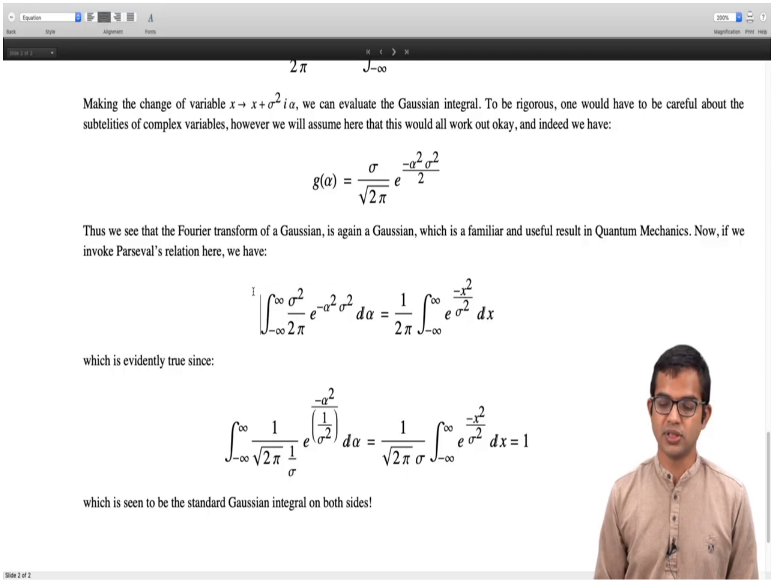

So, you can make a change of variable x to x plus sigma square i alpha, right and then it is just this standard Gaussian integral, right. So, one has to be a bit more careful and you know to be completely correct and you know use the certainties of you know complex variables. However, here we will just assume that it is all good and indeed it is true. So, complex variables are involved.

So, then you have to be a bit more careful, but so, if we assume that it is all good and so, indeed g of alpha is equal to sigma over square root of 2 pi e to the minus alpha squared sigma squared by 2, right. So, this part will just give you 1 over square root of 2 pi sigma and right.

So, this part is going to give you a square root of 2 pi sigma and so, sigma remains and then one of these square roots of 2 pi will cancel and then you have just left with sigma over square root of 2 pi times e to the minus alpha squared sigma squared by 2.

Now, if we invoke so, we have managed to find the Fourier transform if we invoke Parseval's theorem right so, into this relation. So, we immediately see that you get minus infinity to plus infinity sigma squared over 2 pi e to the minus alpha squared sigma squared d alpha, right.

So, this is you know the integral of g of alpha mod of this squared and then you have to put in this factor 1 over 2 pi and that is followed by this integral minus infinity to plus infinity f of x the whole square. So, that is going to become e to the minus x squared by sigma square dx, right.

So, which is evidently true because you know you just recast this in this form right, instead of 1 over 2 pi you write it as 1 over square root of 2 pi on both sides and then you bring in one of these factors sigma to the other side. So, you have 1 over square root of 2 pi sigma here on the right hand side and on the left hand side you rewrite this as 1 over square root of 2 pi 1 over sigma then exponential of minus alpha square divided by 1 over sigma square you know d alpha.

And, then you see that both these integrals are really the same: the left hand side and the right hand side and they are both equal to 1 right. So, they are both the standard Gaussian integral and so in fact, we get a result which is not a surprise, right. So, in quantum mechanics you might have encountered you know Fourier transforms of this kind Gaussian integral particularly is of very great importance.

So, there the physical interpretation is you know it has to do with you know wave function conservation right. So, it does not matter in which basis you expand your wave function right. So, if you sum over all the probabilities of it being in all possible you know eigenvalues corresponding to that operator then it has to add up to 1 right.

So, that is one way of thinking about this if you are looking at it from the point of view of quantum mechanics. But, the result that the Fourier transform of a Gaussian is a Gaussian and therefore, the inverse Fourier transform of a Gaussian is also a Gaussian right is of great importance and it appears in all kinds of context including quantum mechanics. So, that is all for this lecture.

Thank you.