Mathematical Methods 1
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Fourier Transforms
Lecture – 53
Fourier series to Fourier transforms

So, we have seen how given a reasonable periodic function, it is possible to write it as a Fourier series right. So, we first started with functions with period 2 pi and wrote it in terms of cosines and sines and then, we relax this need for the period to be 2 pi, we said ok an arbitrary period can still be you know there is a rescaling that one can do, and it is still some function with an arbitrary period can be expanded in terms of sines and cosines right.

So, the whole prescription has been laid out. So, a natural question which appears is what happens if your function is non-periodic? Right. Is there some notion of you know which is like that of Fourier series which can be generalized for a non-periodic function? Right; so, which is the subject matter of this lecture ok.

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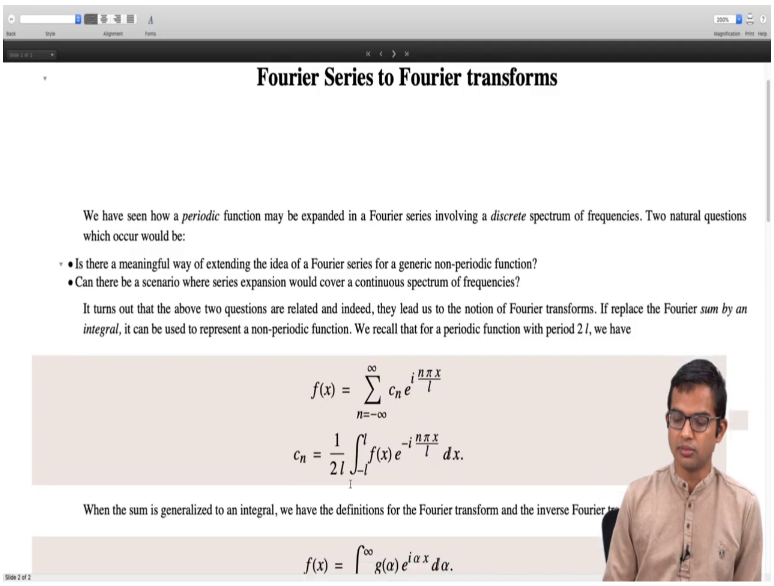

So in fact, two natural questions appear right once we have are familiar with Fourier series, one of them is whether there is a meaningful notion of Fourier series or something like that for a non-periodic function and the other is you know we saw that there is a discrete set of frequencies, we are expanding our periodic function as a sum over a discrete, but infinite set of frequencies right.

But now the question is, is there some scenario in which this sum will become an integral? Meaning that the frequencies allowed would be continuous in nature right, so it is a continuous infinity and not just a discrete infinity right; it turns out that both these questions are related right and their answers are basically the same and so that is the notion of the Fourier transform right.

So, we will describe how you know if you take a limit in an appropriate way, the Fourier sum will become an integral which is which goes by the name of a Fourier transform right. So, let us start by recalling how if you have a periodic function with period 2 l, it can be expanded as a summation over n c n e to the i n pi x over l where these coefficients c n are written down as these integrals 1 over 2 l integral minus l to l f of x e to the minus i n pi x by l dx right.

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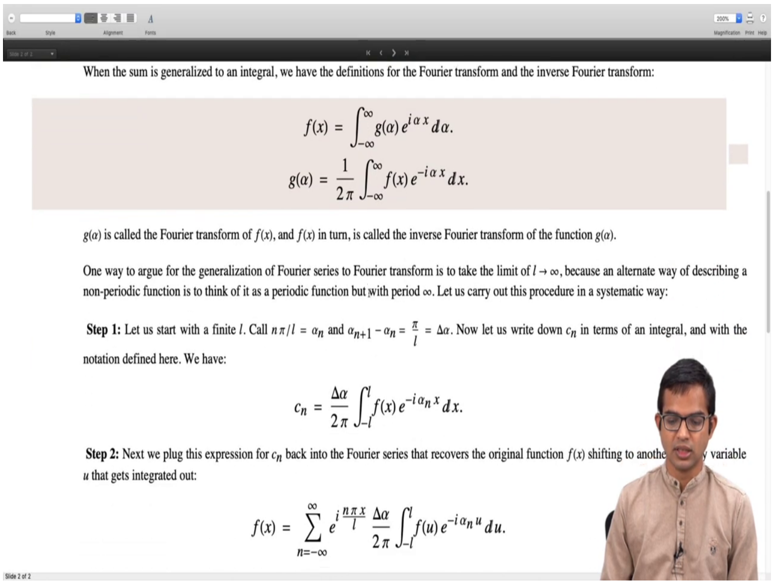

So, when this sum is generalized to an integral, we have you know I yeah, the definitions are modified like here. So, f of x is now a sum right. So, the coefficients are now become a function g of alpha so, this n is gone so, in place of n pi over l, you have an alpha and alpha takes all possible values from minus infinity to plus infinity you know all continuous values, real values from minus infinity to plus infinity and you have g of alpha, e to the i alpha x d alpha and then, g of alpha is you know also it now runs from so, the integral runs from minus infinity to plus infinity right.

So, earlier we had an integral which was restricted to some interval minus l to l right. So, because it has a period 2 l and all the information contained in the function was given within this interval, but now that interval has to go all the way from minus infinity to plus infinity and then, you will see how you get this factor of 1 over 2 pi f of x e to the minus alpha x comes in and then dx right ok.

So, let us see how the generalization of this series to Fourier transform happens right; so, basically what we will do is take the limit l tending to infinity right so, that is what is going on here right. So, let us carry out this procedure in a systematic way.

So, we start with a finite l and then, we redefine n pi by l as alpha n right. So, this n index n is left so, it is just a relabeling exercise at this point and so, then we you know denote alpha n plus 1 minus alpha n which is pi by l as delta alpha right. So, you know you can see that if you are going to be taking l to infinity, this delta alpha will go to this you know differential element d alpha.

So, now, let us write down c n in terms of integral. So, now you have c n is delta alpha over 2 pi right so, in place of 1 over 2 l so now, you see where this 2 pi is coming from right, you sort of see right now, but you will see in a moment in detail minus l to l f of x e to the minus i alpha n x dx.

So, next we plug in this expression for c n back into this summation expression for f of x. So, f of x becomes summation over n going from minus infinity to plus infinity, you have this e to the i n pi x over l, then in place of c n, we have delta alpha divided by 2 pi minus l to l f of u e to the minus i alpha n u du.

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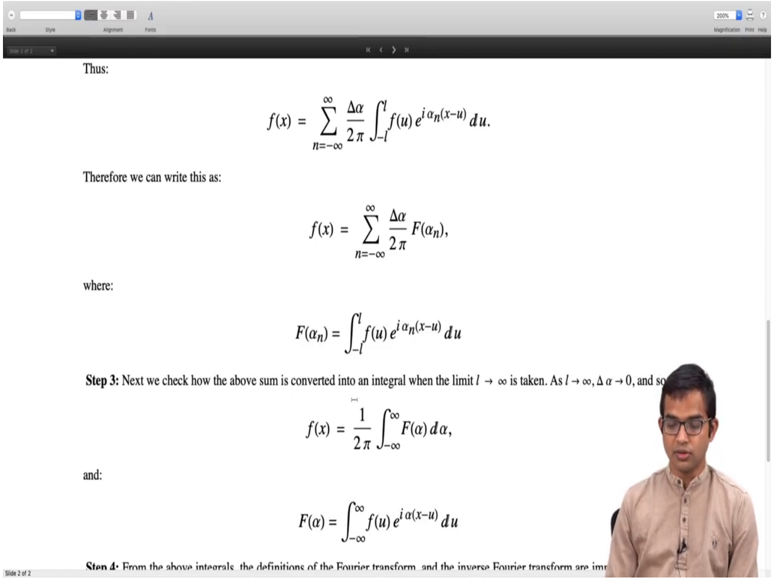

So, we have f of x is equal to summation over n delta alpha over 2 pi integral minus l to l f of u right. So, we bring in this factor exponential i to the pi i times n pi x over l in here and then, we join these two and then we just have e to the i alpha n x minus u right. So, here of course, we have identified n pi by l is just alpha n.

So, now, we have this expression for f of x where there is a sum and an integral and so, thus, we can rewrite the same thing as summation over n delta alpha divided by 2 pi and that is a function of alpha n right.

So, we already see that this is when you are going to take this limit delta alpha to become arbitrarily small so, this summation is going to become an integral. So, where alpha F of alpha n is also given right; so, F of alpha n is this integral minus l to l f of u e to the i alpha n times x minus u du.

So, now we take this limit l tends to infinity so, this summation will just become an integral right. So, in place of F of alpha n, we just write it as F of alpha and then, you get a d alpha comes in because you have delta alpha right. So, 1 over 2 pi of course, remains as it is.

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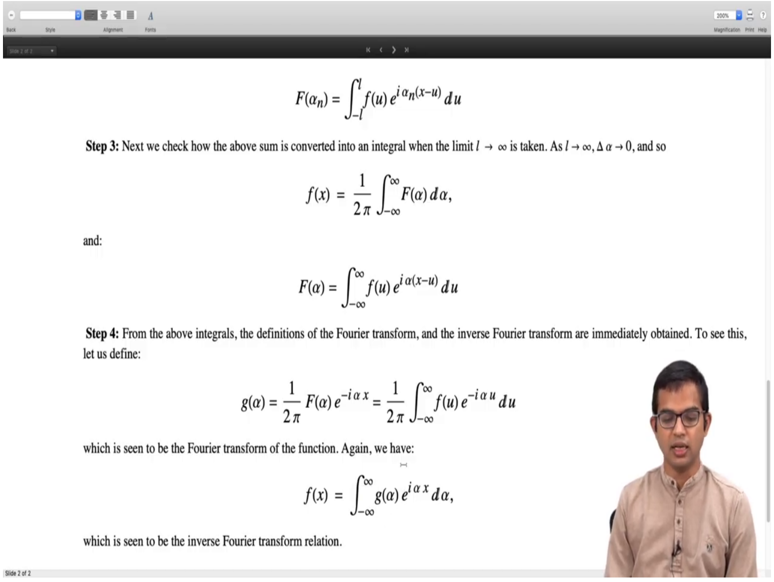

And now, what is F of alpha? Right. So, let us look at what F of alpha is. Now, F of alpha has also become you know an integral going from minus infinity to plus infinity f of u and alpha n should be called alpha because alpha takes all these all continuous values and so, the definitions of the Fourier transform, inverse Fourier transform everything follows immediately right so, but we have to just you know identify these quantities properly.

So, in order to do this, let us define g of alpha is 1 over 2 pi times F of alpha times e to the minus i alpha x. So, then, we see that you have you know 1 over 2 pi is here and then F of alpha is here right so, in place of F of alpha, you plug in you know so, this integral and then this factor basically you know cancels one of these factors and then you are just left with integral minus infinity to plus infinity f of u times e to the minus i alpha u du right. So, u is a dummy variable right.

So, it is convenient to use u here and anyway it is getting integrated out and then, there is also this overall factor of 1 over 2 pi. So, which is really the same as the definition for g of alpha that we have used right except that in place of x so, here we had f of x and you know e to the minus i alpha x dx, but here we have u in place of that. So, it does not matter whether you integrate over u or whether you integrate over x as long as you are consistent with the way you are doing it.

So, g of alpha is called the Fourier transform of this function f of x where, f of x itself is written in terms of this Fourier coefficients right if you wish you can think of g this g of alphas as Fourier coefficient, but actually it is called a Fourier transform right and now, it is an integral and it tends from minus infinity to plus infinity and g of alpha times e to the i alpha x d alpha right. So, this is the idea of Fourier transform which is the generalization of the notion of a Fourier series. That is all for this lecture.

Thank you.