Mathematical Methods 1

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Fourier Series
Lecture – 51
Differentiating Fourier series

Ok. Hello, so, we have been looking at Fourier Series. So, we have given the prescription for writing down Fourier series for a function which is periodic. We have also seen what the Dirichlet conditions are which guarantee convergence. So, we will look at the issue of what happens if you took a Fourier series of a function and if you are going to differentiate it, is it legal to differentiate it term by term right and when is it; when is it legal right that is the question which we will address in this lecture ok.

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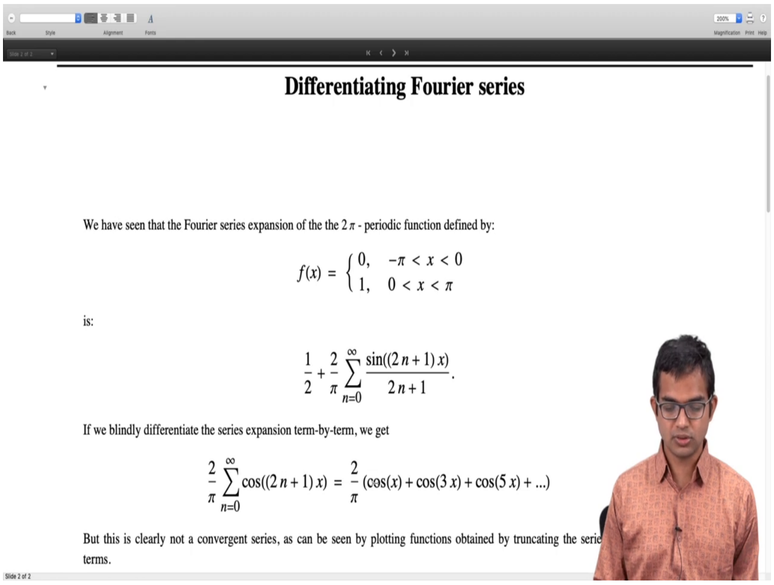

So, let us look at an example right. So, this is the example which we have looked at repeatedly so, let us take this example, f of x equal to 0 from minus pi to 0 and this is function 1 from 0 to pi and we have seen that this has the Fourier series representation half plus 2 by pi summation over n you know all the odd terms will appear sin of 2 n plus 1 times x divided by 2 n plus 1 right. So, we saw that this nicely converges and indeed because Dirichlet conditions hold right.

So, but it is if we blindly differentiate this term by term right so, we could go ahead and do it right. So, if we the series that we will get the new series after differentiation, you see that you know sin of 2 n plus 1 x when you differentiate it, you will get this 2 n plus 1 which will come out and cancel with the 2 n plus 1 in the denominator right.

So, you see that these coefficients as n becomes larger and larger are dying down because you have this you know 1 over 2 n plus 1 factor right. So, I mean if you think of it as in general as it goes as 1 over n right the coefficient n.

Now, but when you differentiate it, you know this cushion of dying down factors does not exist and so, you get a series like this 2 by pi cosine of x plus cosine 3 x plus cosine 5 x. So, on and it is not hard to see that this is actually not going to converge right. So, we can quickly verify this by plotting a sequence of partial sums.

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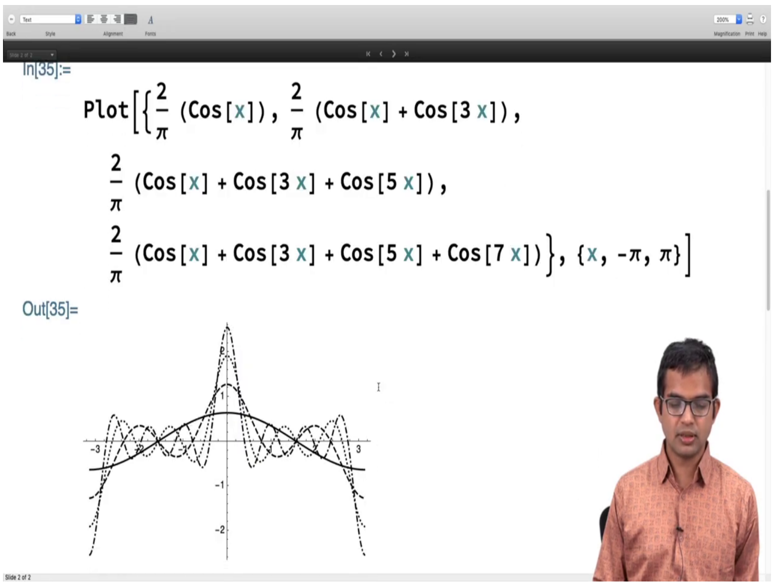

So, let us look at this. So, I have plotted here you know just the first term, then the first term plus second term and first term plus second plus third you know for one, two, three, four yeah you know these four different partial sums if I plot it, this is what it looks like.

So, you see that you know there is you know evidently there is a lot of mess at the origin, but also at pi and minus pi and also even at you know other points, you see a lot of you know craziness, it is not really you know as the number of terms are increased, it is not going to some nice in a you know not a converging value right.

So, there may be certain special points where you know you may be able to get some interesting you know special results at special points, but the key idea here is that this is not a convergent series, as you can also see from just looking at this function and so, why is this happening?

It is not a surprise right because if you look at it, you know some thought reveals that there is mess at the origin and at x equal to 1 right. So, differentiating the series is reflecting the mess that appears when you differentiate this function itself right.

So, it is not a problem if you differentiate in the region minus pi to 0, but at x equal to 0 there is a big problem and at x equal to 1 also there is a problem right because of the jump. So, it is not a legitimate you know operation to do, to do a term-by-term differentiation.

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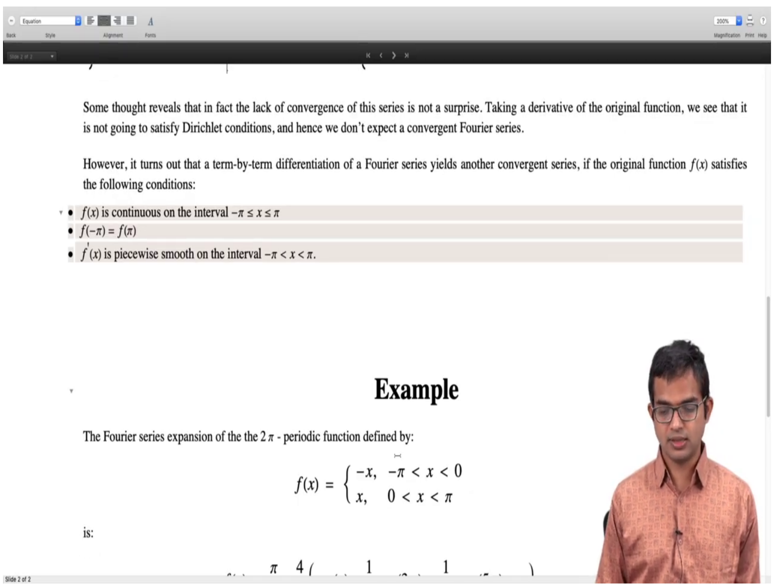

So, Dirichlet conditions by themselves are not enough to guarantee a meaningful result if you do a term-by-term differentiation. So, but however, it turns out that you know term by term differentiation is allowed if a function is you know satisfies stronger conditions, it is not enough if it is Dirichlet, but in fact, you if your function if it is continuous, it is not enough if it is piecewise continuous, the whole function if it is continuous in the entire interval from minus pi to plus pi including at the end point.

So, f of minus pi should be equal to f of pi and the derivative should be piecewise smooth right. After all, if you want a you know legitimate Fourier series for f prime right so, then this is the condition. So, you would expect that f prime itself should be extremely well behaved after all you are going to be seeking a Fourier series for f prime of x.

And so, let us look at just one more example of this kind. So, if I take f of x equal to minus x you know in the region minus pi to 0 and x in the region 0; 0 to pi, then indeed you know all these conditions hold, you can plot this function it is a continuous function and it is its derivative is also well behaved as you will see.

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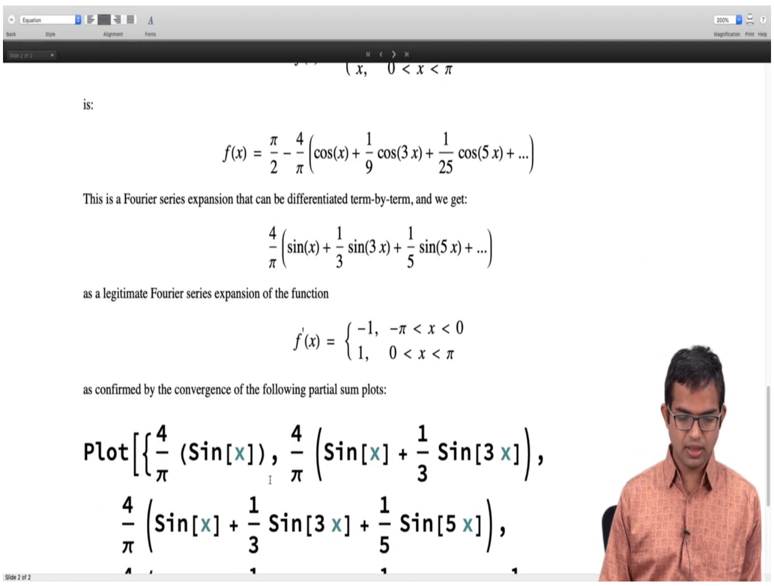

So, if you if I work out the; if I work out the Fourier series so, now, you see I have pi by 2 minus 4 over pi cos of x plus 1 over n cos 3 x plus 1 over 2; 25 cos of 5 x so on. So, now, you notice that these coefficients are not just going as 1 over n, but in fact, they seem to be going as 1 over n square.

So, when I take a derivative, I still have you know 1 over n's which appear. So, of course, cosines will become sines right, so that is not a problem right. So, an even function will become an odd function and vice versa when you; when you know take the derivative that is not an issue.

But what is key here, what has changed from the previous example is that now you have this cushion of 1 over n you know factors of 1 over n you know appear in so, as your coefficients become larger and larger, you know the higher order terms you know mean less and less. So, you can expect that this is going to converge and indeed it is true right.

So, you can check that this is the same Fourier series you would get if you had you know started with this function f prime this itself if you took your as your function and worked out the Fourier series you will get this. So, this is a legitimate case where taking a derivative of the Fourier series term by term gives you a meaningful result. As you can also check by plotting it right at the convergence of the partial sums is also something that we can verify right.

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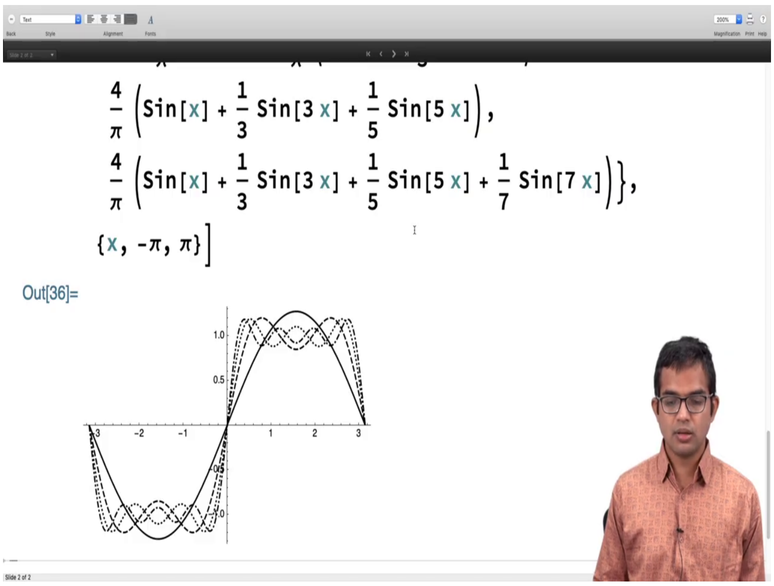

So, I am just looking at the first maybe four terms, you know just the first term, then truncating after the second term, after the third time, after the fourth term and I find that indeed eventually this is going to go to this you know the step function right, it is you know this function minus 1 from minus pi to 0 and it is a square wave actually right. So, it is going to be periodic.

So, the point of this discussion was to you know is to tell you that you should be careful if you are going to take a derivative of this function and yeah, so, if your function is you know if it can be written as a sinusoidal Fourier series right so that means, your function is not only periodic, but it is also it is also odd right so, then yeah, so, then you will have to demand that it goes to 0 at both ends right.

So, sin of pi is 0 and sin of minus pi is 0 so, your function itself its value must; must go to 0 at both ends right. So, you can work out what the special conditions are if you say that your function is not only periodic, but also even or if it is odd, both cases you can work out. If it is even, then it is going to be cosines right. So, those are special cases.

And you might ask what happens if we integrate the Fourier series? Integration is a little more robust than differentiation. So, if your original function is represented by a valid Fourier series, you can integrate you know both sides and it is a meaningful relationship for both sides right.

There are two pieces of caution, one must exercise, one is of course, there is a free constant of integration which comes up right. When you are integrating the left-hand side and you are also integrating the right-hand side so, there are free constants on both sides, you have to fix both these constants to be the same right. So, that is kind of a trivial thing, but it is essential to take care.

The other is that what you get may not be a Fourier series right. So, suppose you have something like a naught you know you start with a constant and then, you have you know all these terms in sines and cosines, the constant part when you do an integration, it is going to give you a naught times x now, that is not a Fourier series representation right. So, everybody else you know sines will become cosine and cosines will become sines, but you will also have this one extra x.

So, if you had started with a function which did not have this a naught, then what you get after integration would be a Fourier series representation of some other function right of the integral, but if you have a you know a constant here so, when you take the integral in fact, so the function is not going to be periodic because this is not a Fourier series, but it is a valid expression right.

So, this is just a comment which I am making on the side, but this discussion was primarily about you know differentiating Fourier series and you know saying that one should be careful right, it is possible to do term by term only if your condition, your function you know is extra smooth, it must satisfy these conditions that we have laid down ok. That is all for this lecture.

Thank you.