Mathematical Methods 1
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Fourier Series
Lecture – 50
Even and Odd Functions

Ok. So, now, we have laid down the prescription for a function of arbitrary period, you know reasonably well behaved function periodic in nature, which can be written as a Fourier Series. And, we have also looked at the Dirichlet conditions, which guarantee convergence. So, it turns out that if a function is either Even or Odd some extra simplifications arise right; so, which is a topic for this lecture ok.

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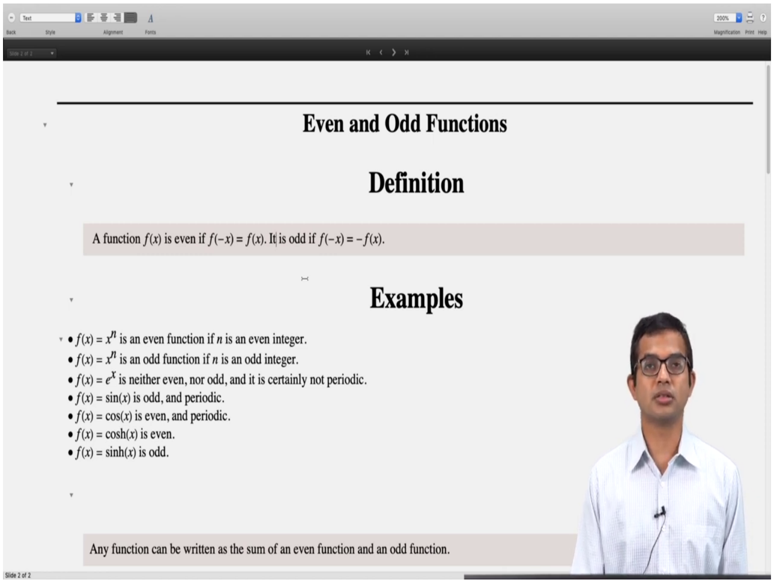

So, a function is even if f of minus x is equal to f of x it is odd, if f of minus x is equal to minus f of x right. So, it is a familiar concept so, let us just very quickly look at a few examples, if f of x is equal to x to the n. Then, it is an even function if n is an even integer, and it is an odd function if n is an odd integer right. You can verify this by plugging in the condition.

Now, if f of x is equal to e to the x, then it is not even, it is not odd, it is not periodic right it is a strictly increasing function right e to the x. And on the other hand if f of x is sin x, then it is odd and not only odd, but it is also periodic right. So, this is the type of you know function which we are particularly interested in this discussion right.

So, or f of x is equal to cos x is even and it is also periodic right. You know hyperbolic cosine function is even hyperbolic sine function is odd right, neither of these will be periodic, but there they have oddness or evenness.

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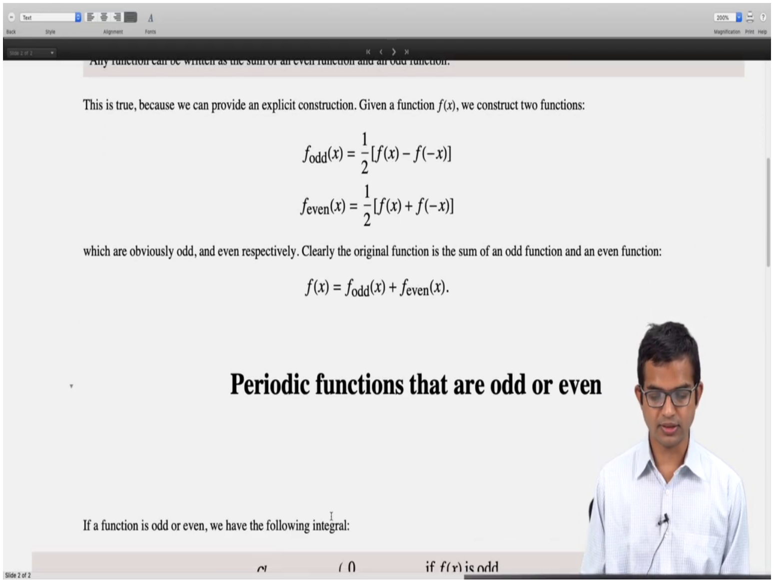

Now, any function can be written as a sum of an even function and an odd function right, you may want to pause the video and very quickly provide a construction for how this is true right on your own right. Or look at the argument that I am going to give you this is true because, evidently you can just give a construction for it right.

So, consider a function f of x and then you know build these two functions, one of them is I am going to call it f odd of x half of f of x minus f of minus x. And, f even of x which is half of f of x plus f of minus x. And of course, not every function is either even or odd right, this we have already seen examples of functions which are neither even nor odd, other examples you can construct on your own.

But, given any function f of x, you can consider these combinations and you can evidently see that the first of these functions is odd. And, the second of this is even right, you simply put minus x in place of x. Then, you will see that the first function there is going to be an extra overall minus sign, which will come out in relation to the original function; whereas, the second function nothing changes if you change x to minus x.

So, clearly the original function itself is the sum of these two right, you can you know immediately see this. So, indeed any function f of x can be written as the sum of an odd function and an even function right. So, if you have some periodic function f of x right.

So, you know working out the Fourier series of it you know is a linear operation right. So in fact, you can work out the Fourier series separately for the odd part and for the even part. Oftentimes this act of separating on the odd part and the even part is not going to give you any you know reduction in labor.

But, sometimes it gives you some extra insight right because you see that cosines and sines, have their evenness and oddness respectively right. So in fact, we will see that if a function is given to you that is odd and periodic, then it has properties right. Let us look at periodic functions that are odd or even, if they are then you have some simplifications.

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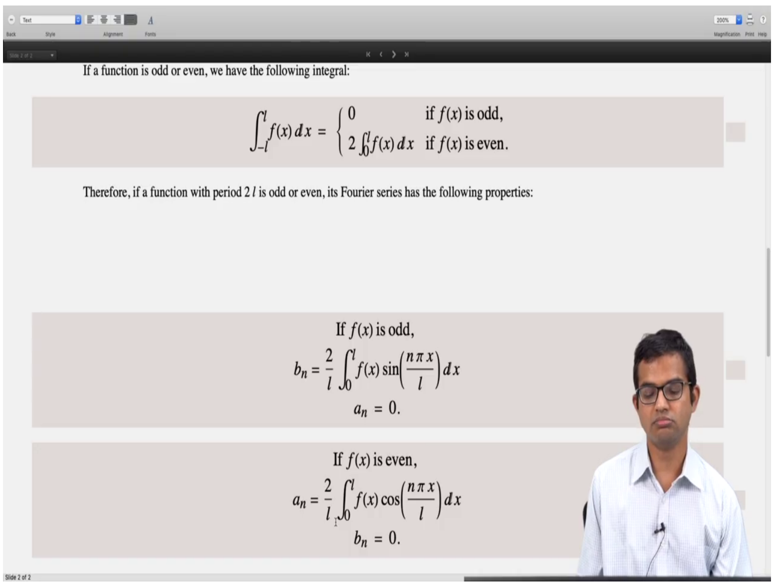

So, if a function is odd or even then we have this following integral right, it does not matter in which interval you go from minus l to plus l, if your function is odd then this integral is going to vanish right evidently. Because, you just use the relation that f of plus x is equal to minus f of minus x.

And, then you break this interval from minus l to 0 and another interval from 0 to l, you know sum these two and then you can quickly convince yourself that you know these two parts will vanish. And, an even better way to do this is to actually do it graphically just plot it you know plot a function which is sort of the mirror image of itself you know to the left of the origin and, but with a negative sign.

And, then you will see that the area that comes in on the right from positive x will cancel with the area that comes in from negative x. But, on the other hand if f of x is even you do not have to do the full integral from minus l to l you do, you know one of these parts which is let us say from 0 to l and then you just take twice of this right.

So, that is a consequence of the evenness right. So, this result would be true no matter which interval that you consider. But, if you are looking at a periodic function with period 2 l right so, it is natural to look at this entire interval for you know, which considers one full period of this function.

Then, the Fourier series we have seen you know it has integrals of this kind. So, it will lead to the following simplification, if f of x is odd then all the a ns will vanish right. So, indeed, you cannot have any you know representation from cosines, if your function is odd, you cannot have any cosine part of it. It is going to have only it is going to be made up of a bunch of signs.

And likewise if a function of f of x is even, then it is all the b n are going to vanish and the a n are going to be given by this integral right 2 by l. Now, the interval you need to consider is 0 to l right.

So, this is actually a direct consequence of this result. I urge you to use this result and in the prescription given for computing these coefficients for functions with period 2 l. And, then show for yourself that indeed these two results hold when your function is not only periodic, but it also has oddness or evenness associated with it ok.

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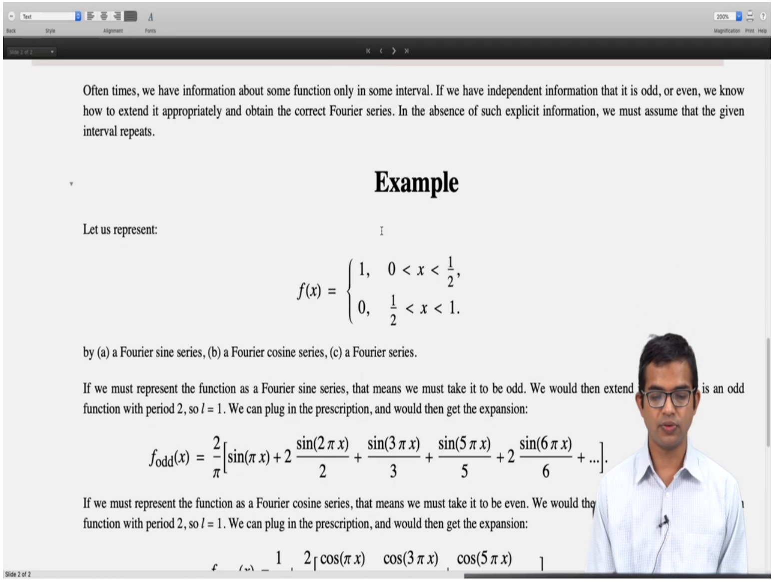

So, we had this discussion about piecewise continuous functions and piecewise smooth functions and so on right. There was I in at that point it was just we are looking at form some functions, which is defined in some interval a to b right. And, it is not necessary that you know there is any periodicity about this.

But, given information about a function in some interval, we can extend this function such that the function is odd. Or you may extend this function such that it's even, or sometimes it makes sense to just extend this function as is given whatever information is given, take that itself as the period and repeat it right.

So, you know there are contexts where you know each of these is an actual thing to do. So, let us look at an example. So, suppose you are just given this function f of x in this interval right f of x is 1 from 0 to half and it is 0 from half to 1. Now, if you are asked to expand this or extend this function such that you should be able to give a Fourier sine series for this.

Then, you would extend it in a certain way right such that, it is going to be an odd function right whereas, if it is a if you want to Fourier cosine series you know to represent its extension. Then, you would extend it such that it is an even function, or if you know if you just want to construct a Fourier series. And assuming that you know this is one full period of this function, then that is the third kind right.

So, let us look at all these three and you see how you know the Fourier expansion would change. If we must represent this function as a Fourier sine series that means, we must take it to be odd right. And the interval is the period is 2 right; so, from minus 1 to plus 1 right, whatever information is given is given from 0 to 1 you must impose oddness to this. And so, you say that you also have information for the value of this function from minus 1 to 0.

So, imposing oddness immediately gives you information about the value of this function from minus 1 to 0. And, then you take this as your unit which repeats right. So, if you do this and then carefully work out the prescriptions which are given here, then you will find that you know the odd extension of this function is 2 by pi you know this whole expansion.

So, you can make a plot of this function and concentrate on the interval, you know 0 to 1 it is going to agree exactly with this. You know outside of that it's going to get represented in a different way, depending upon whether you have extended this function to be you know into a Fourier sine series or if you do it in a cosine series, then you must make it to be an odd function right.

So, now you will get a different series, but if you take a close look at only the interval of interest, then it does not matter which of these you are looking at. They all give you the correct answer right. So, you need some input which is independent of this information, on whether you want to extend your function to be an odd function or an even function or you know just you know take this itself as the period which is the third kind right.

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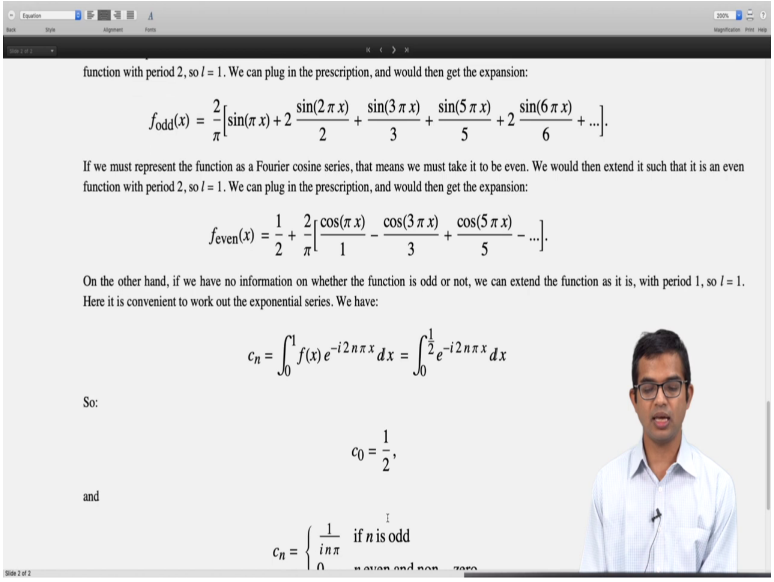

So, if you take it directly as a period then in fact, it is you know it is convenient to just look at the exponential series right. So, then you have c n is given by this expression and then like, I said it is always good to take special care for c 0 when so, c 0 you have to compute separately and then you get a half. And then c n is just 1 over i and pi if n is odd and it is 0 if n is even and non zero, if it is 0 then it is half otherwise it is 0 right.

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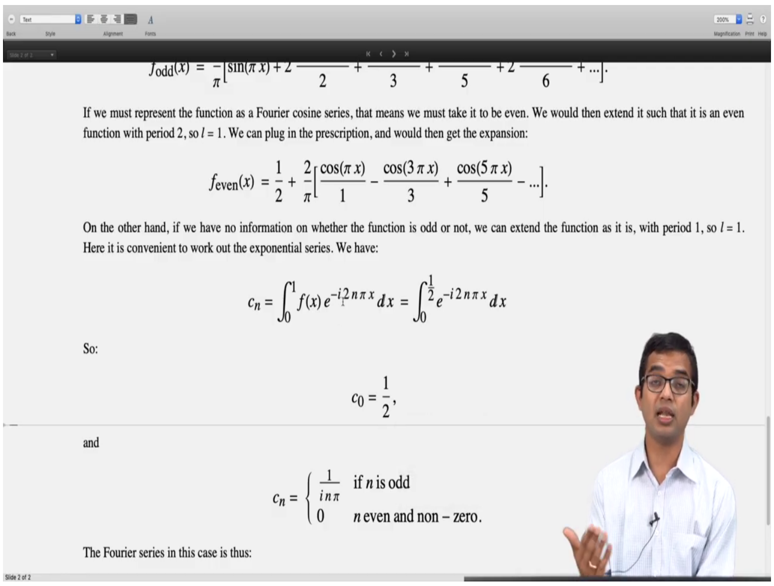

So, the Fourier series in this case is so, you notice all these you know twos here which comes from this n pi by l right. In this case l is a half right whereas, in the first two cases l is actually

1 so, meaning this entire information from 0 to 1, you have to take a copy of this and paste it on the left of the origin right. So, minus 1 to 1 will be your period. Whereas, here in the third case you just take 0 to 1 itself as your period and so in fact, l in this case is 1.

So, it is in fact l equal to l l is equal to half. So, the period is 1 so, 2 l is 1. So, l is equal to; l is equal to half right. So, here it is convenient to work out this exponential series and then we have you know this expression involving minus i 2 n pi x d x you have to be careful. And, then the final result because of these you know i n pi x by l in this case l is a half. So, you get all these factors of 2 and you have only these odd ns are allowed.

So, you will see that this exponential series is going to be something like 1 over 2 plus 1 over i pi e to the 2 i pi x minus e to the minus 2 i pi x plus 1 over 3 e to the i 6 pi x minus one third e to the minus i 6 pi x and so on right; so, it would be an instructive exercise for you to take you know truncated versions of each of these series and plot that right, using some plotting software it does not matter which one right use whatever is convenient for you.

And, then check that indeed all of them you know basically agree in this interval 0 to 1 right. Because, they are all valid extensions of the same function, but you have all you have input some extra ingredient in working out these different representations for this function right; these are they are different extensions of this function, ok. That is all for this lecture.

Thank you.