Mathematical Methods 1
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Linear Algebra
Lecture - 05
Inner product on a linear vector spaces

So, we saw how one can work with the dot product when we are dealing with Euclidean vectors right, it was mostly in the nature of recall. Now, we are going to look at how some of these ideas can be generalized. So, we will introduce the notion of a scalar product of vectors in a linear vector space ok.

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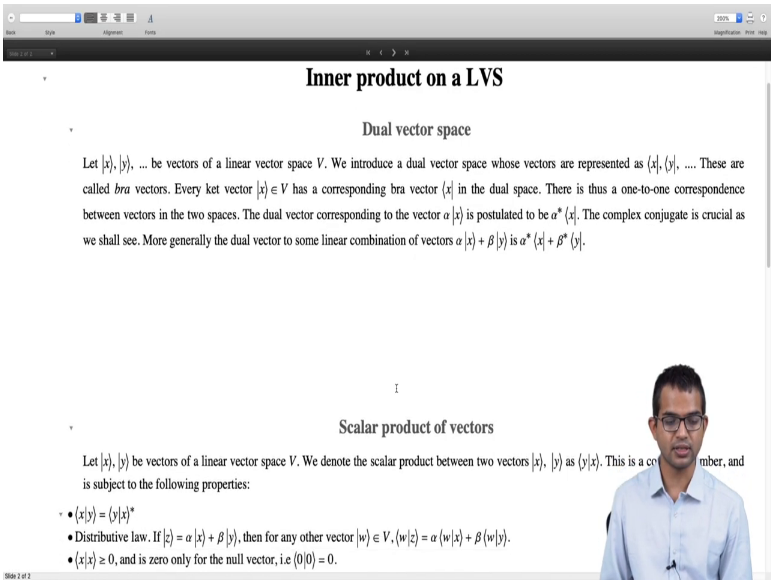

So, before we do that let us point out that there is something called a dual vector space. So, whenever you have a vector when you define a linear vector space, automatically a dual vector space gets defined right.

So, you have x, y and so on. Let them be you know these ket vectors as we call them, they form some linear vector space V. So, the notation you know these are called bra vectors right. So, these angular brackets are pointing in the left direction right. So, these vectors, so for every ket vector, there is also a bra vector right which belongs in a dual space.

Now, so there is one-to-one correspondence between vectors in the two spaces and so importantly whenever you take some vector in the ket space and multiply it by a complex number right, so of course, the basic requirements of you know it being a vector space implies that this alpha times x is also a vector which lies in the same space.

Now, the vector corresponding to this vector alpha x which lies in the dual space is going to be alpha star of this bra vector x right. So, this conjugate, this complex conjugate is an important part of this you know setup right. So, we will see how this plays out as we go along.

So, in general, if you have some linear combination of ket vectors alpha x plus beta y, the dual vector corresponding to this is going to be alpha star bra x plus beta star bra y vector right. So, we will use this sort of at an operational level right; we do not make it any more abstract than it needs to be.

So, let us think of this as a you know as a given, there is a dual space where all these vectors are defined, and then we will see how you know the notion of a scalar product that we will introduce now is intimately connected to this dual vector space right. I mean it will appear later on in many of the manipulations that we will do. So, we will learn to work with this space – this dual vector space ok.

So, how so we said that the dot product of you know two Euclidean vectors right, we wrote down their properties. So, based on those properties, we are going to come up with a prescription for a scalar product between two vectors. So, let us say that you know x and y are two ket vectors which belong in some linear vector space V. And we want to find the scalar product between these two vectors x and y.

And here it is denoted by this you know this bracket notation you know this inner product is denoted as you know the bra vector y acting on the ket vector x. So, this is in fact where the nomenclature of bra bra and ket comes from because it looks like a bracket. And so Dirac said so we will look at vectors which you know appear on the left side we will call them bra vectors, and those vectors which appear on the right side we will call them ket vectors right. So, this is where the nomenclature comes from.

Now, we define this scalar product of these two vectors to be some complex number subject to the following properties. First if you take the scalar product of x with y, it is going to be related to the scalar product of the vector y with x, but they are not exactly the same in general.

So, in fact, we are going to only require that one of them is the complex conjugate of the other right. So, we will see in a moment why you know it is not possible to make it a you know harder constraint, and make it completely symmetric like it was possible with you know Euclidean vectors right.

So, the scalar product of x with y is going to be the scalar product of y with x you know with a complex conjugate operation performed on top of this and the distributive law is very important. So, we will require that. So, if z is sum you know alpha x plus beta y and then if you are going to take the scalar product with w right with the bra w coming in, so you get this expression. So, w x z is equal to alpha w x plus beta times the inner product between w and y like here.

And then we also have this very important requirement that the inner product of any vector with itself must be greater than or equal to 0 right. So, but so I mean we just said that you know the inner product of two vectors is going to be a complex number. So, does it even make sense to demand something like greater than or equal to 0 right? So, complex numbers are not ordered. You cannot say things like one complex number is greater than something else right.

So, you can only talk of this if this quantity is a real number. So, in fact, we will show you know immediately we will show that already we have put in enough constraints that the inner product of a vector with itself is automatically real right. First of all its real already follows from what we have said so far. And in addition to it being real, we also require that it must be greater than or equal to 0 right.

Why do we do this? So, this is because it will allow us to think of or to define the notion of a you know the equivalent of a length for a vector. So, we have seen that with 3D vector or

Euclidean vectors, you can take the dot product of a vector with itself, and this has the interpretation of its magnitude square right.

So, we want something like that here and that is only possible if this quantity is positive definite right, and it is going to be zero only for the null vector right. So, these are all the three requirements for some quantity to be a scalar product right. So, there is actually more than one way in which you could define a scalar product for a you know set of vectors in a vector space.

As long as these three properties are satisfied, it is an acceptable scalar product ok. So, now let me give you the argument for why you know we cannot make this symmetric, we cannot hardwire a requirement that the scalar product of x with y is the same as the scalar product of y with x right because these are complex numbers right.

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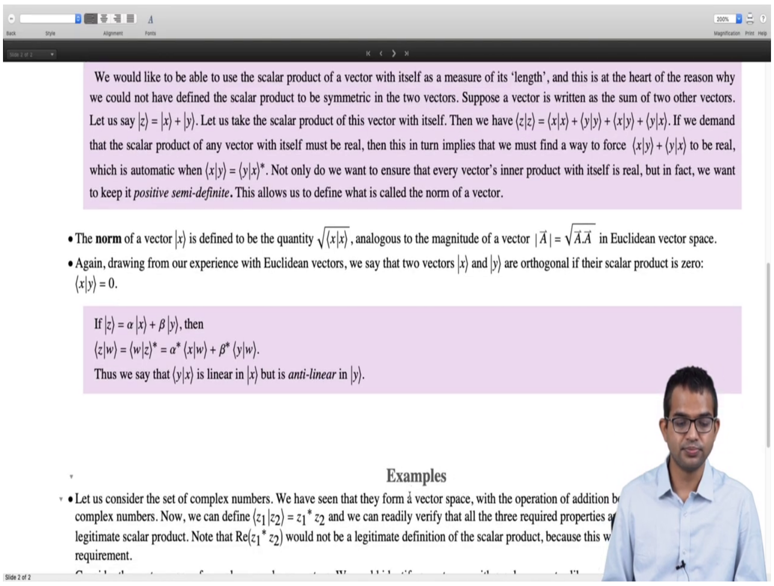

So, and so you know the main reason is that we would like the inner product of a vector with itself to be a real number right, and not only a real number it should be greater than or equal to 0; and that can be forced only if you have this complex conjugate right. So, let us look at the argument for this.

Suppose, you have a vector z which is the linear combination of two vectors x, x and y right; you could have some coefficients there, but it does not matter let us just take it to be x plus y.

Now, if I take an inner product of z with itself, then the inner product of z with z is equal to the inner product of x with x plus inner product of y with y plus inner product of x with y plus inner product of y with x.

So, now if I demand that each of these quantities: the inner product of z with z, x with x, and y with y, they are all real right from you know this idea of its magnitude to be encoded in these quantities. They all have to be real. If I demand this, then it forces this quantity the inner product of x with y plus the inner product of y with x to be real.

And this will not be compatible with a symmetric definition. If I just demanded that x y must be equal to y x and this quantity should be real that will happen only if the inner product of x with y will be a real number right, so that is not going to be sufficient right. In general this inner product is going to be a complex number. So, there are certain you know special vector spaces right – the real vector space where that can happen right. We will look at such cases.

And in which case anyway the complex conjugate of the real number will just reduce to itself, so; there is no issue there coming in the other direction. But, in general if you have a complex number for this to be you know compatible with every inner product of any vector with itself to be real, we must you know it will work out if you demand you know x the inner product of x with y is equal to the complex conjugate of the inner product of y with x right, so that is.

So, if this condition holds, then automatically you are guaranteed that the inner product of any vector with itself is going to be real. And then on top of that we also have this requirement that it must be greater than or equal to 0. Why do we do that is because it will allow us to define something called the norm of a vector right so which is what I said it is like the magnitude of a vector.

So, the norm of a vector x is defined to be this quantity square root of the inner product of this vector with itself it is completely analogous to your modulus of a vector A in Euclidean space. Now, again drawing from our experience with three-dimensional vectors, we will say that two vectors in this you know abstract linear vector space; they are orthogonal to each other if the scalar product between them is 0 right.

So, and also you know something which follows from all this discussion so far is if you have some arbitrary linear combination alpha x plus beta y for some ket vector z, then if you were to take the inner product of z with some other ket vector w right with you know the z vector going into the brass space.

So, then you have z w the inner product z w is the same as the inner product w z with a conjugate complex conjugate. On top of it which is the same as alpha star right, so this alpha star is very important. And this is something that you know you practice with these quantities and then it becomes automatic, but you know when you are seeing it for the first time then this is a place where people get confused.

So, pay attention to when you go from the ket space to the bras space. So, there are always these conjugates for coefficients involved right you have to be careful with. So, alpha star x w plus beta star y w and so we say that you know this inner product y x is linear in x, but it is actually anti linear in y. So, this antilinear simply means that you have to do these complex conjugates when there are these coefficients involved ok.

So, let us look at a few examples. So, I said that as long as these three properties hold, you know you can define your complex number corresponding to any two vectors in more than one way. So, let us look at how for some vector spaces that we have already seen, how we can come up with the idea of a scalar product.

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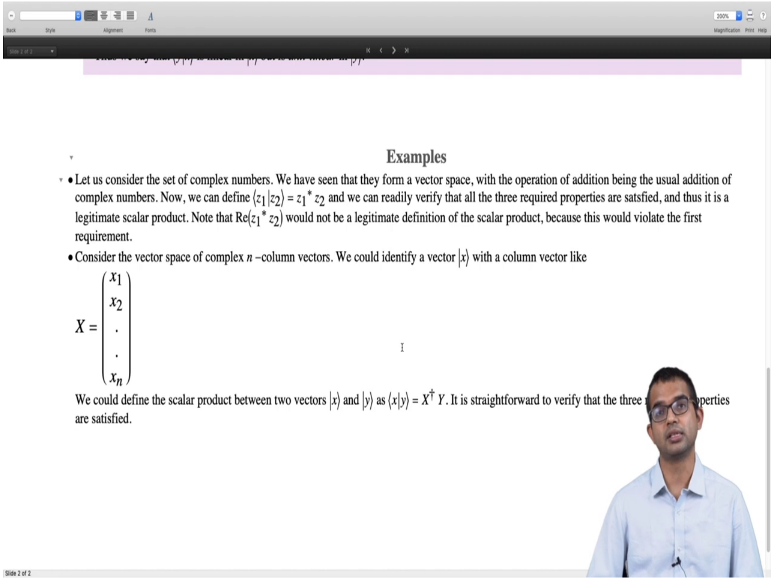

So, let us consider the set of complex numbers itself. I said it is a vector space right. We have seen that they form a vector space with the operation of addition being the usual addition of complex numbers. So, now, we can define. So, the ket vector z 1 is simply something that you associate with a complex number z 1 right.

And so if you find if you define the inner product of two complex numbers you know the vectors corresponding to two complex numbers z 1 and z 2 like here, so I can define it as z 1 star times z 2. And we can readily verify that all these properties are satisfied you know first of all so if I take z 2, z 1, of course, it is going to become the complex conjugate.

So, that is good. Then I can look at the inner product of you know this vector with itself that is going to be just z 1 star z 1 that is going to be modulus of z 1 squared which is a positive definite quantity, so that is a check. And, also we have the third which is the distributed law which also holds right all readily verified. So, this is a legitimate scalar product.

Note that we could have tried to do something more complicated like try to define the scalar product as the real part of z 1 star z 2. And this is not going to work out right. So, the reason is if you take the inner product of z 1 and z 2, it is going to be a real number here, and then its complex conjugate needs to be the inner product of z 2 z 1. And this is not going to work out. You can check this right.

So, not everything which will be some function of z 1 and z 2 or you know of two vectors that you can think of will be a legitimate definition for a scalar product right. There is definitely more than one way of doing it, there are multiple ways of doing it, but you have to carefully check this and then you have to be consistent with your discussion right.

So, many a time that kind of vector spaces we will deal with the notion of scalar product will be that will be a natural definition for it right like with Euclidean vectors right. We have a natural definition. But also there also one can come up with other ways of defining a scalar product right.

So, we might look at some examples at a later time, but I am just telling you that the definition of scalar product does not constrain it to be just you know one particular function there are more than one there is more than one way of doing it as long as those three properties hold.

So, let us look at complex n column vectors. We could identify a vector x you know with a column vector like this - you have n complex numbers. These are all the coefficients. And then we could define the scalar product of two vectors x and y as the inner product of x y is equal to x dagger y right. So, this bra vector can be thought of as a column vector right. So, whenever you have a ket vector it is like you know a column vector, whereas, the bra vector is a row vector right.

So, there is a column vector as far as y is concerned whereas a bra vector is a row vector right so that is why you have this x dagger operation. And it is straightforward to verify that you know all these properties required by inner product hold. So, that is all for now in this lecture. We will see some consequences of these as we go along starting from the next lecture.

Thank you.