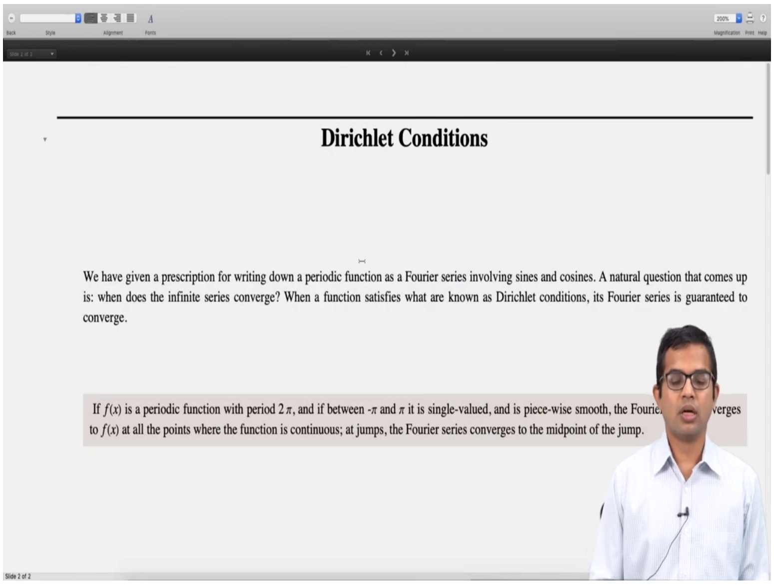
Mathematical Methods 1

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Fourier Series
Lecture – 47
Dirichlet Conditions

So we have looked at how a periodic function with period 2 pi can be expressed as a Fourier Series. So, in this lecture, we will look at issues of convergence. So, it turns out that if you know, if the function satisfies certain you know very reasonable properties, then you know the series is indeed going to converge right, so that is the content for this lecture right.

(Refer Slide Time: 00:51)



So, the Dirichlet conditions would guarantee that you know the Fourier series corresponding to a periodic function would indeed converge. So, it is a theorem right. So, what we are going to state now is you know has a rigorous theorem status, but of course, we are not going to prove it.

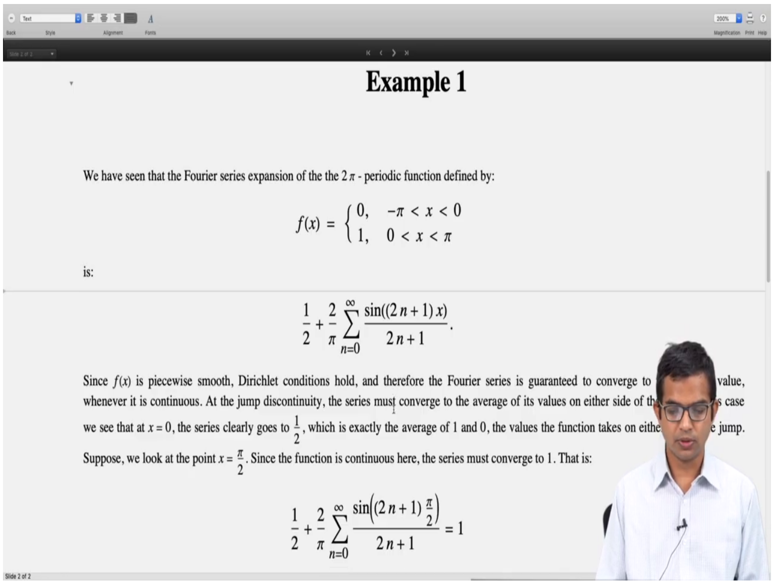
We are going to just you know state what the Dirichlet conditions are right. So, we have seen that it is useful to think of a piecewise smooth function right. So, there is a reason why we introduced that notion and that has to do with the Dirichlet condition right.

So, if your function is periodic with period 2 pi and if between minus pi and pi, of course, it has to be single valued and it is also piecewise smooth. So, what is a piecewise smooth function? A piecewise smooth function is one which is piecewise continuous and whose derivative is also piecewise continuous.

And if both these conditions hold, then the Fourier series is guaranteed to converge to f of x at all points where the function is continuous, there is no ambiguity right. So, the Fourier series also converges to the value of the function at that point.

But when you have these jump discontinuities that is where you have some ambiguity for what the value of the Fourier series would be at that point, and it turns out that the Fourier series as we have you know written down is going to is going to converge to the midpoint of the jump right. So, that is f of x naught plus plus f of x naught minus divided the whole thing divided by 2 right. So, these Dirichlet conditions have very nice applications. Some of which we will see in this lecture ok.

(Refer Slide Time: 02:52)



So, let us look at an example. So, we have seen that the Fourier expansion of this 2 pi periodic function. It is useful to look at an example that we have already seen, f of x equal to 0 in the region minus pi to 0, and it is equal to 1 if you know x lies between 0 and pi. Now, for this function, we have already seen that the Fourier series expansion is given by this quantity right.

So, now, we are worried about what this means right? It is an infinite series, whether you know whether this converges to this function right, but you know we have already seen evidence of this convergence right. So, because you can just plot this function to you know truncating at higher and higher levels with keeping more and more terms. And we see that the more terms you keep, the closer this seems to get to the original function itself right, so that is evidence for the fact that it is going to converge right.

So, although I mean there is this function that is rigorously going to converge to this because it satisfies the Dirichlet condition. So, indeed this function f of x is piecewise smooth right, it is piecewise continuous and its derivative is also piecewise continuous. So, indeed, it is piecewise smooth. And therefore, this Fourier series is guaranteed to converge to the function value wherever it is continuous. And if it is a jump discontinuity the series must converge to the average of its values right.

So, you can see that the only point of difficulty with you know this kind of a function is at x equal to 0 and if you put x equal to 0, all these sins are going to go to 0 right. Every one of the terms in this series other than this constant term everybody else is going to go to 0. So, you see that evidently the series is going to go to half right when you put x equal to 0.

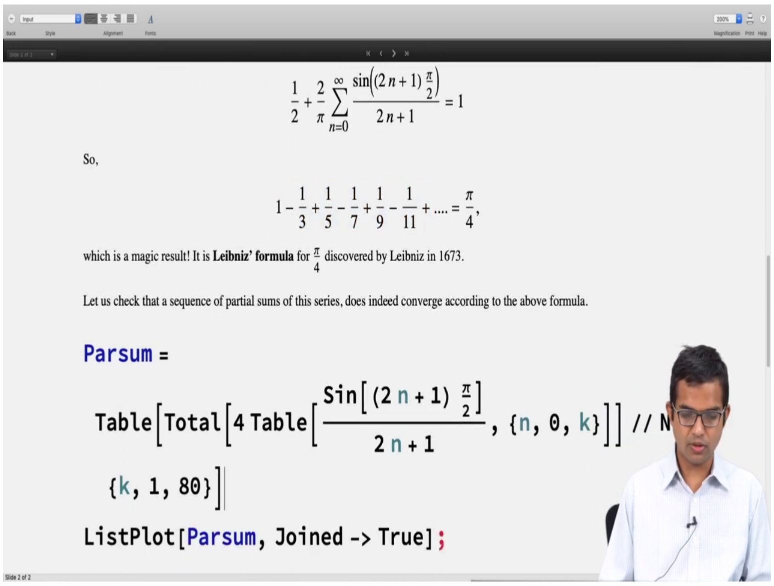
Whereas, the function itself we have not even defined what it is at x equal to 0 right. So, you just like the Dirichlet theorem, you know told us it told us that at a jump discontinuity the series is going to converge to half the value of the function you know slightly to the right of the discontinuity and slightly to the left of the discontinuity which is indeed true. Now, let us look at you know we can plot this and check that the two are becoming better and better which we have already done.

And I urge you also to come up with your own plotting. Use some plotting software and truncate this series at various orders, and then check that it is becoming better and better as you keep more and more terms right.

So, now let us just pick one point. If I take x equal to pi by 2 right, so between 0 and pi, it is supposed to go to 1 right so if I just pick one value x equal to pi by 2. So, now, the function is continuous here.

So, Dirichlet theorem tells us that the series must converge to 1 right, so that means we should be able to say half plus you know 2 by pi summation n equal to 0 to infinity. So, where I have x in place of x, I put pi by 2; and on the right hand side, I am going to say that this is equal to 1 right. This is a consequence of Dirichlet theorem, and the fact that this function is continuous at x equal pi by 2.

(Refer Slide Time: 06:14)



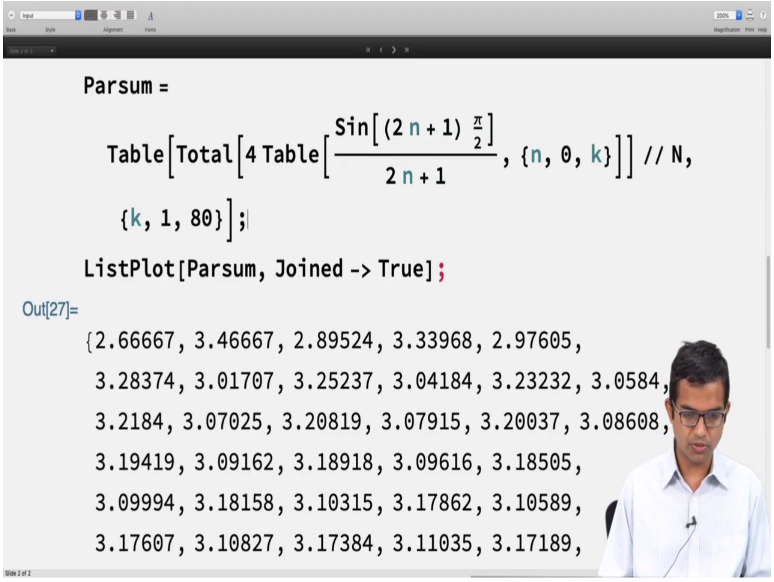
So, then I you know pull this one to a half to the other side, and then I take this factor by 2 by pi and multiply throughout by pi by 2. So, then I have and I write this out explicitly the series right. So, I have 2 n plus 1 pi by 2. So, it is going to be pi sin of pi by 2, sin of 3 pi by 2, sin of 5 pi by 2 and so on. So, you see that there is going to be an alternating sine which comes in. So, I have 1 over 1 minus 1 over 3 plus 1 over 5 minus 1 over 7 so on.

So, this series quite remarkably you know what this theorem and Fourier series analysis tells us is that this c series is going to converge to pi by 4 right. So, it is quite a remarkable result which seems to have been pulled out of nowhere you know and since it is like a magic trick. So, it turns out that this is what is called the Leibniz formula right for pi by 4. And it was discovered by Leibniz in 1673 right. So, it turns out that you know Dirichlet conditions can yield many interesting results for us.

So, if we have not already looked at this convergence of these functions like I have a you know a series of plots here right. So, if you have already done it, that is fine; or I urge you to do it yourself, but I also have a sequence of plots right. So, I have, well I am using some mathematical notation; if you are not familiar with it, it's not a problem.

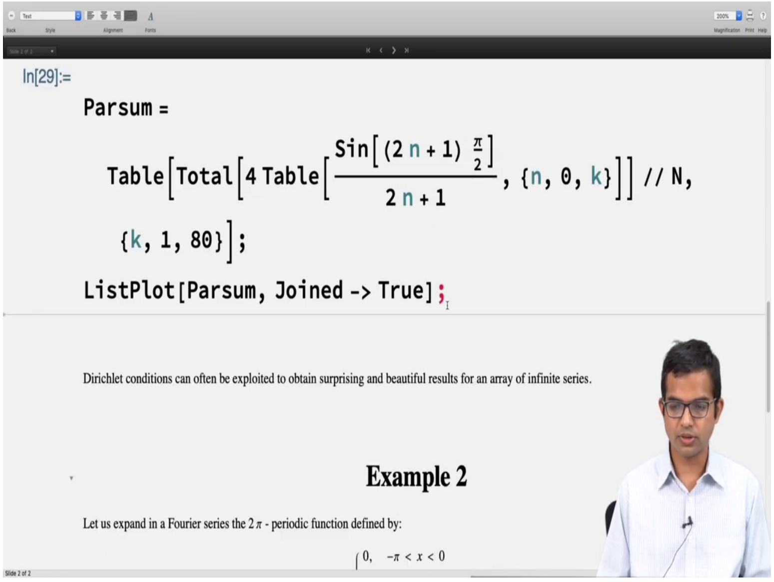
So, what I am doing is I am taking you know this summation sin of 2 n plus 1 times pi by 2 divided by 2 n plus 1, and you know truncating it at various levels. And then multiplying throughout by 4 because I want to verify just this, but look at 4 times this to verify that it is going to go to pi right.

(Refer Slide Time: 08:31)

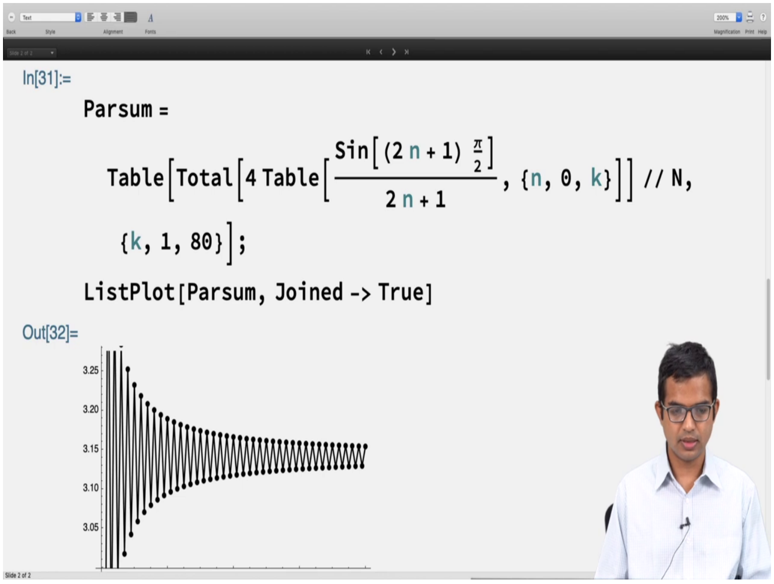


So, let us look at this. I am going to just not worry about giving you the details of what the code is, but let me just show you the plot.

(Refer Slide Time: 08:38)



(Refer Slide Time: 08:42)



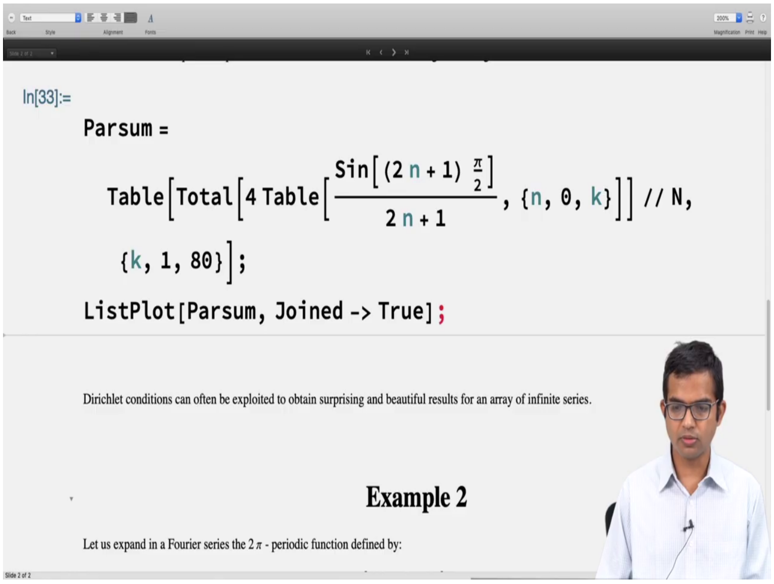
So, if I show you the plot it is going to look like this. There you go. So, you see that as I keep more and more terms in this series, I am just summing this right to higher and higher orders. I find that it is going to get closer and closer to pi by 4. And it is an alternating sequence - these partial sums you know they approach pi, no doubt, but you know they miss pi by an amount which is which alternates in the sin right.

So, and that is something that is not surprising because you have you know every term in this sequence is you know as a consecutive term in the sequence changes by a sine right.

So, there you overcorrect for it a little bit, and then you have to go back in the other direction, and then you go in the other direction and so on right.

So, every term that you add will cause you to miss pi by an amount which is either positive or negative depending upon which time you are at right. So, that is also cleanly captured in this plot right.

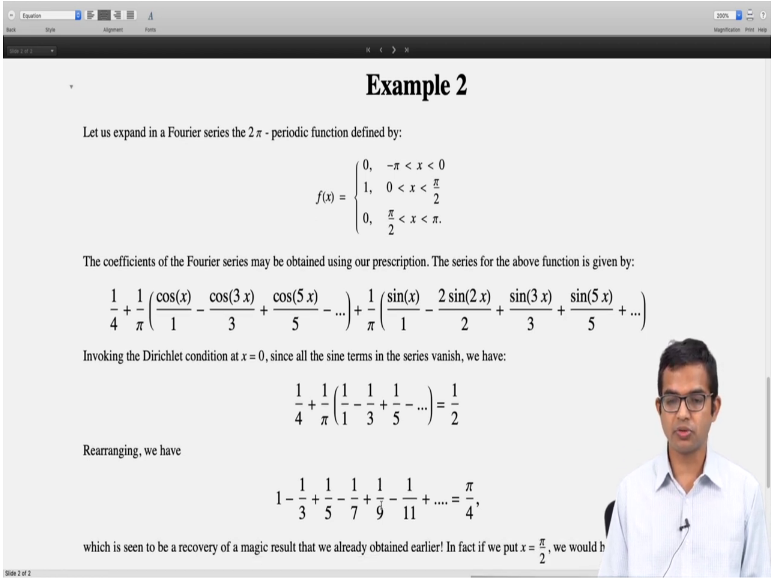
(Refer Slide Time: 09:53)



So, it is a confirmation that indeed this, this you know very beautiful result holds numerically we are able to you know see that it is a plausible result right. So, it's quite remarkable that you have all these you know rational numbers on the left hand side.

But if you know add an infinite number of these rational numbers, it can go to pi right ok. So, now, Dirichlet conditions can often be exploited to obtain surprising and beautiful results, many such infinite series can be pulled out, many such magic tricks are possible.

(Refer Slide Time: 10:27)



So, let us look at another example. So, let us expand in a Fourier series this 2 pi periodic function defined by you know f of x equal to 0 from minus pi to 0, and it is one only in an interval from 0 to pi by 2 right. It is a little different from the previous example, and then again it becomes 0 from pi by 2 to pi.

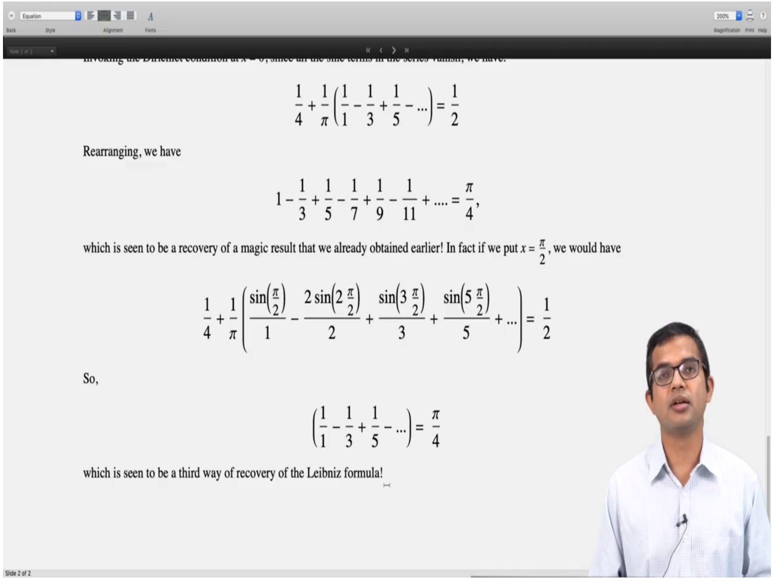
Now, the coefficients of this Fourier series you can obtain right. We have given the prescription for this. So, let us just assume that we know how to work this out right. I am giving you the answer here. So, I have 1 by 4 plus there is going to be a cosine series and a sine series right.

So, invoking the Dirichlet condition at x equal to 0, all the sin terms are going to go right. They do not contribute, because you are putting x equal to 0. And then the cosine terms will give us exactly this same series, in fact, 1 by 4 plus 1 over pi times 1 minus 1 by 3 plus 1 by 5 minus so on, and that this must be equal to half right because Dirichlet told us this.

Dirichlet condition at x equal to 0 guarantees that this series must converge to half this value to the slide of the sum of the values slightly to the right and to the left of this point and that is just half. So, if I arrange terms, then it tells me that 1 minus one-third plus 1 by 5 so on.

This series converges to pi by 4 right recovering the magic result. So, in fact, let us see what would have happened if you put x equal to pi by 2 that is another point of discontinuity.

(Refer Slide Time: 11:56)



We would have 1 by 4 plus 1 by pi right. So, now, all the cosine terms would go. So, then we are left with only the sine part. Then if you carefully look at all these terms, you will find that in fact even here you get 1 minus 1 by 3 plus 1 by 5 minus 1 by minus 1 by 3 plus 1 by 5 minus 1 by 7 so on. So, in fact, it is exactly the same series that comes in if you also look at x equal to pi by 2. And this too confirms that it is going to go to pi by 4. You should check this carefully right.

So, what we have managed to show is that we have stated the Dirichlet conditions and we have told you when the series will converge and when to what value. And we have also given you a few examples showing how some you know very beautiful infinite series results can be obtained with the help of you know Dirichlet conditions applied in a clever manner right. So, that is all for this lecture.

Thank you.