Mathematical Methods 1
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Fourier Series
Lecture – 45
Orthogonal basis: Fourier series

So, we have seen how the set of piecewise continuous functions on some interval, a comma b, forms a vector space. So, in this lecture, we will see how there is a natural basis for this kind of a vector space. So, we will fix our interval to be minus pi into plus pi. Later on we will look at how this can be generalized on the intervals.

And then, we will see how you know naturally from this sort of linear algebra perspective we would get to the Fourier Series. And in fact, in this lecture we will provide a prescription for how we can take a function which is periodic and we will be able to expand in terms of a Fourier series.

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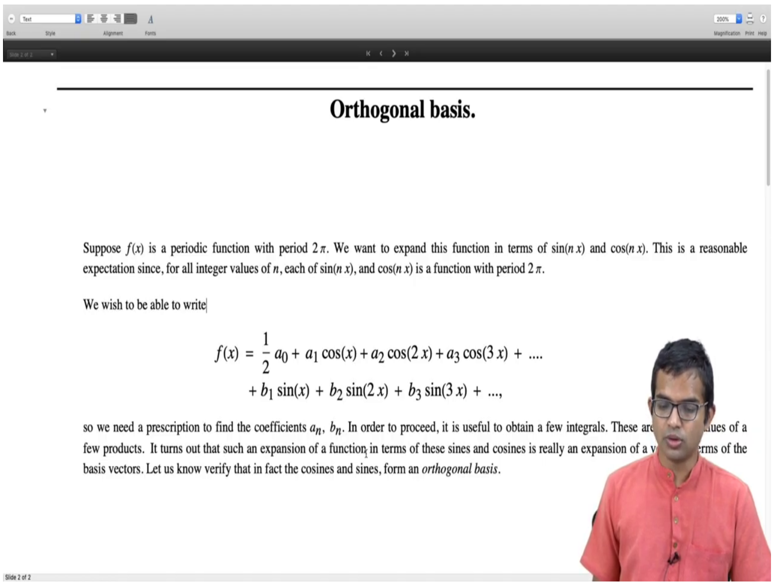

So, suppose f of x is a periodic function with period 2 pi, we want to be able to expand this function in terms of sine of n of x and cosine of n of x, right. So, this is a reasonable expectation. Since, for all integer values we know that sin n x and cos n x they form you know their functions with period 2 pi, but in fact, we will argue that these functions in fact form a basis and they are not only a basis, but they are an orthogonal basis, right.

So, our goal is to be able to write this function f of x as you know half a naught plus it is; well I am calling this as half a naught for a reason which will become clear later on, when we write down the prescription for these coefficients you know. We want to see if there is a way to work out these coefficients half a naught plus a 1 cos x, so on you know if you, so that if we will be able to write this series expansion for this function.

Given a periodic function f of x we will be able to write down you know an expansion of this kind, right. So, we need a prescription to work this out. So, in order to proceed, let us first work out a few integrals, right. So, the periodicity is needed because I mean we are going to think of this function as you know stretching out for all values of x, but in fact all the action is in just one period.

We have seen that you know the average of a periodic function is the same no matter in which interval or equal to the you know period length it is taken, right. So, in that sense, we would be able to focus all our attention on some one period and all the information about the function is known.

So, let us see how you know cosines and sines form an orthogonal basis for you know piecewise continuous functions in the interval minus pi to plus pi. (Refer Time: 03:10) You know it is some period of length 2 pi is good enough, but let us say we look at the region from minus pi to plus pi for completeness, right.

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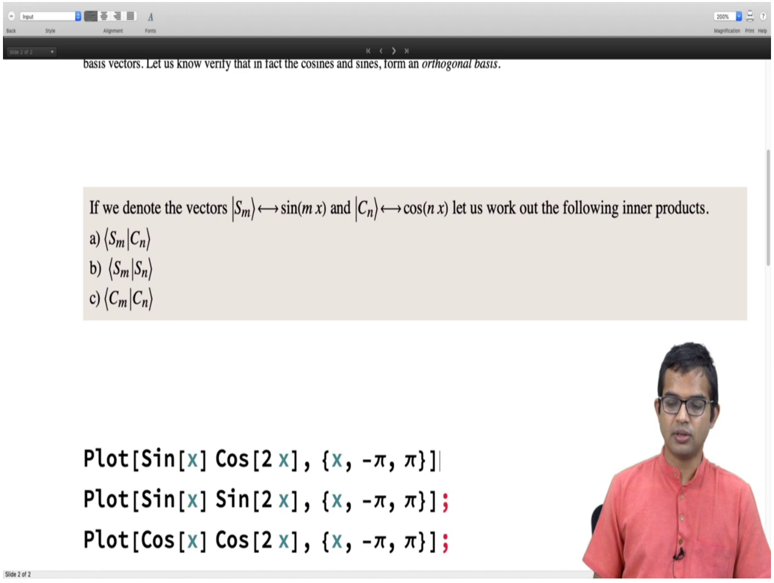

So, if we denote the vectors you know S m as sine of m x and C n as cosine of n x. So, let us work out the following inner products. So, we have already defined this inner product as integral, right and so, we will work out these inner products, right. So, before we do that let us plot a few functions. Suppose, if I plot sine x into cosine 2 x in the interval minus pi to plus pi, it is going to look like this.

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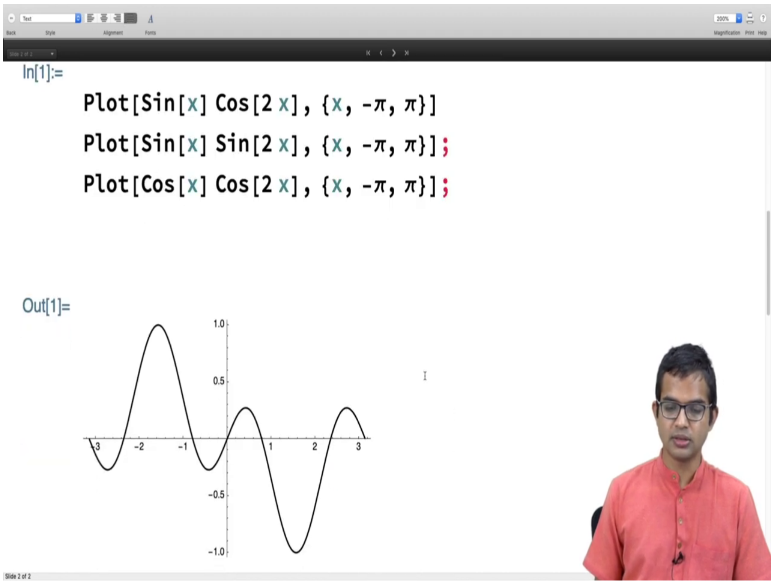

So, you see I am interested in finding out this integral from here all the way up to here. So, you know if you look closely then you see that you know this function is symmetric, right for you know anti-symmetric if you wish, right. So, it spends equal time above as it does below and in a mirror image manner. So, just looking at this visually you would expect that you know this integral is actually going to go to 0.

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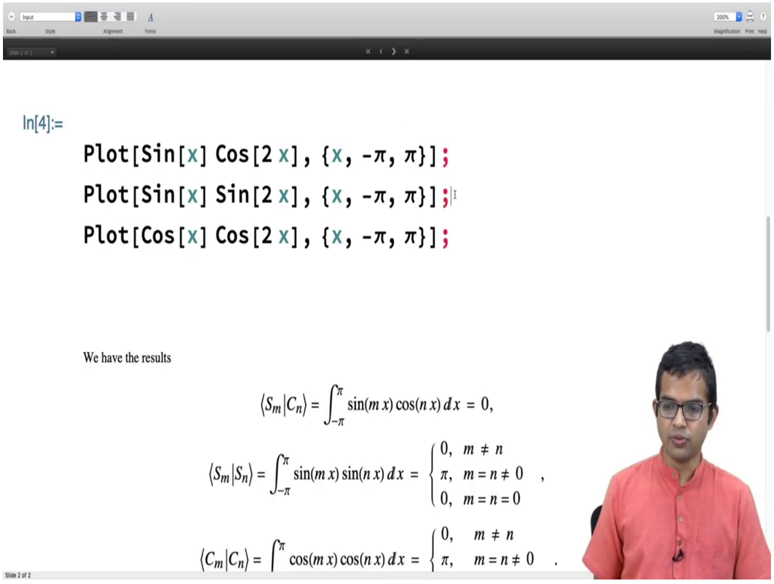

And in fact, that is true as we will argue, and that is also true for cosine sine x times sine 2 x.

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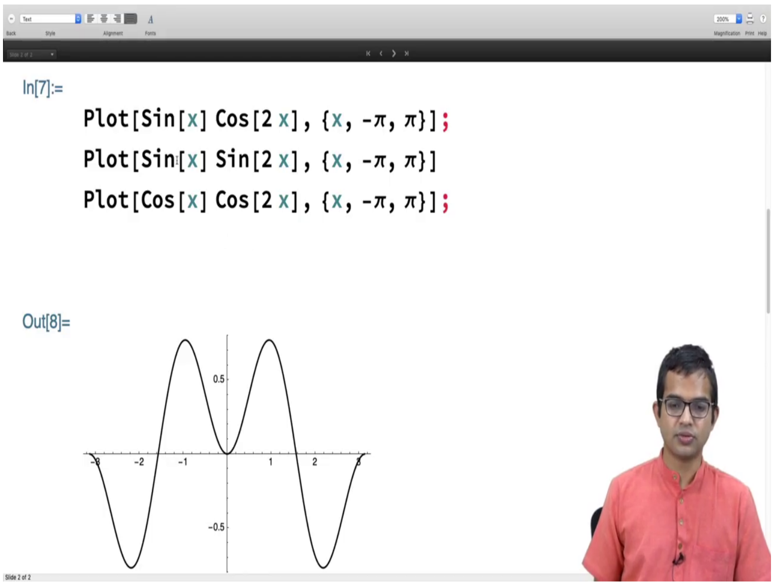

If I take the product of a sine with another sine with you know two different integers here, one here and two here you can try play this game with many other you know functions instead of sine x and sine 2 x, you can try out how the function looks, if you do sine 10 x times sine 15 x you know whatever you want. And then you will see that indeed again you know this part will cancel with this. If I were to integrate this entire period this part will cancel with this and this is going to be 0.

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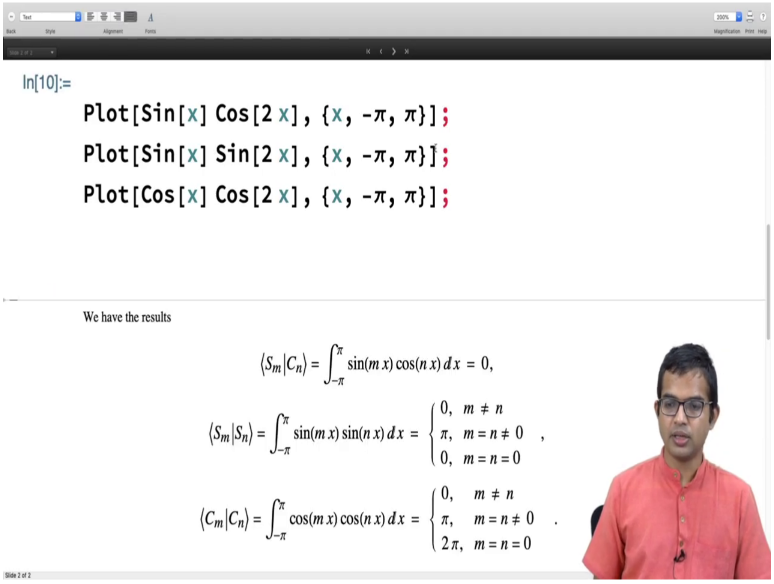

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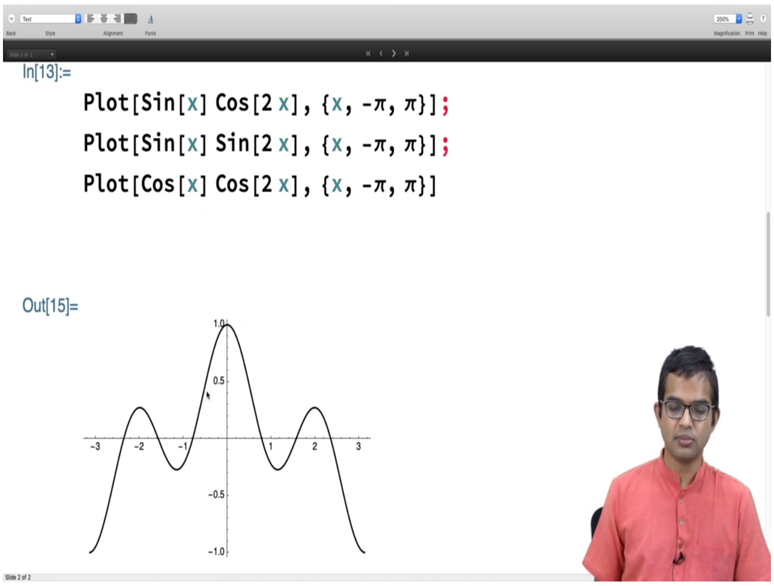

So, and likewise what about if I look at cosine and cosine; cosine x times cosine 2 x, once again you will see that you know there is a way to consider mirror images. So, this is some part whatever part which lies above that part you can make this go away with this part, and then there is an up you know this part will go away with this and this part will go away with this. So, if you were to integrate this, you would expect that this integral of function also will give you 0.

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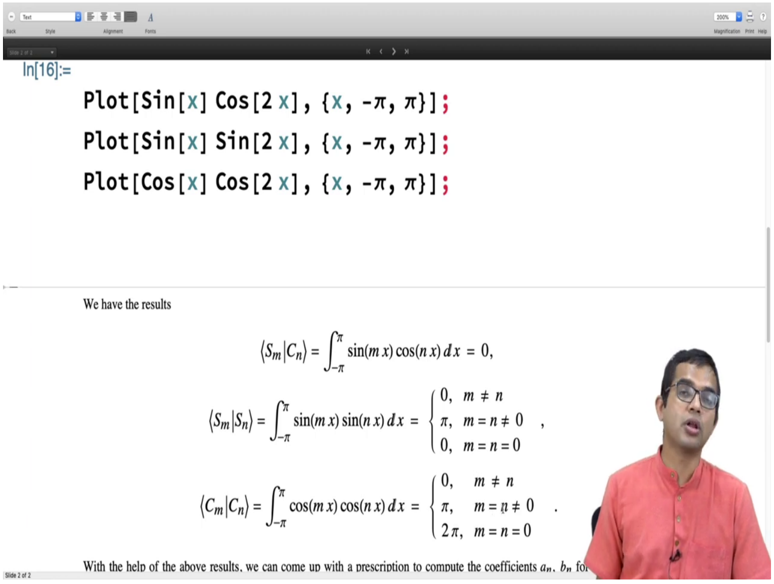

So, in fact, you can use trigonometric identities and show the following results which will be homework. I will allow you to work this out using trigonometric identities, right. So, I have just shown you pictorially or visually how this would play out, but you must show it in a more rigorous way.

So, if I take the inner product S m with C n, I have to do minus pi to plus pi sine of m x cosine of n x dx, it is going to be 0, no matter what m is, no matter what n is. But this S m, S n you know this integral this inner product is going to be 0, if m is not equal to 0 or if m equal to 0 and m equal to 0 equal to 0, right.

So, then it is obviously going to be 0 because you are integrating just 0. But on the other hand if m equal to n you will get a pi, right. So, this will in fact, you know go back to the integral sine squared which we did a few lectures ago.

And likewise when you do the integral of cosine of m x times cosine n x dx, the inner product of C m with C n, you would get a 0 if m is not equal to n. But you would get you know if m equal to n again there are two further cases you have to make. If m equal to n equal to 0, then you are integrating 1 from minus pi to plus pi, so it will be 2 pi.

But on another hand, if you are integrating cos squared x dx for example, let us say then you it is the same as sin squared x dx and then you will get a pi, if m equal to n not equal to 0 then you get a pi, right. All of this is for you to workout. We are all using trigonometric identities directly.

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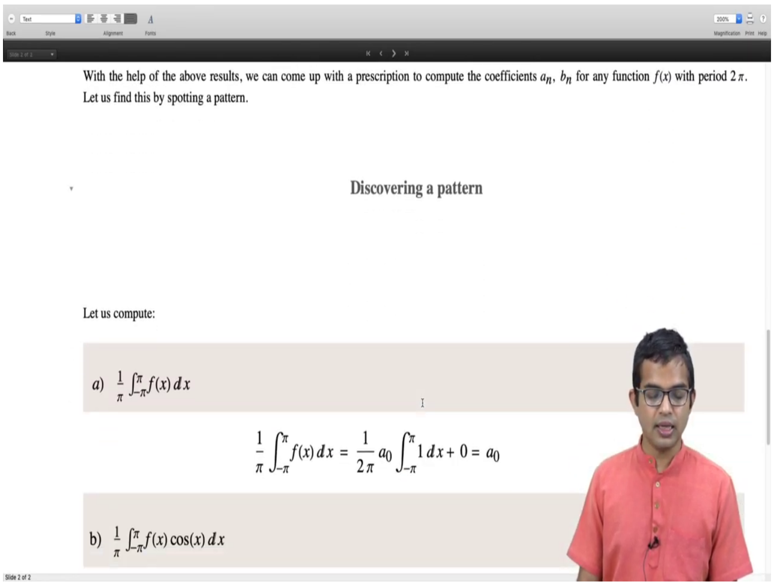

So, the point of what I am doing here is the following. So, there are two things which I am trying to say, right. One is that if I consider a function which is periodic, right, so I am going to restrict my attention to just one period from minus pi to plus pi.

And within this period you know this function is a piecewise continuous function and it is a vector which is a part of this space, this vector space that I have said, right. So, it is profitable to think of this vector space perspective, right. Although, we could have just you know gone ahead and done the algebra and worked this out directly from you know just trigonometric identities and so on.

I am just telling you that you know what is going on here is the following. There is also a vector space way of thinking about this. So, in fact, all these cosine x's and sine x's according to what we have just shown are orthogonal to each other, right.

So, we can also make it orthonormal, you know then you will have to make you know redefine these vectors with some factors. So, let us not do that. So, let us leave this pi's and 2 pi's and so on. Let them be there. But the key point is that if you take any two of these vectors and take the inner product of them then you get 0. So, they are all orthogonal. And so, what is the meaning of this?

The fact that you can take any vector which belongs to this space and expand it in terms of all of this is an indication that this is a basis. So, in fact, this is an orthogonal basis for this space and that is what Fourier series really is, right. So, let us work out these coefficients now, you know by looking at a pattern. So, if we have this, if we compute minus pi to pi 1 over pi f of x dx, then, what happens?

So, we have you know this expansion on the right hand side, because of the above result we see that since all of these functions on the right hand side, cosines integrated in one period, sines integrated in one period, they all give you 0. And so, the only term which will remain is the constant term which will give you just a naught.

So, in fact, a naught is this the average of your function you know subject to this you know there is a factor of 2 which we have introduced, right. So, you have to be careful. So, basically the point is that the constant term is the average of your function.

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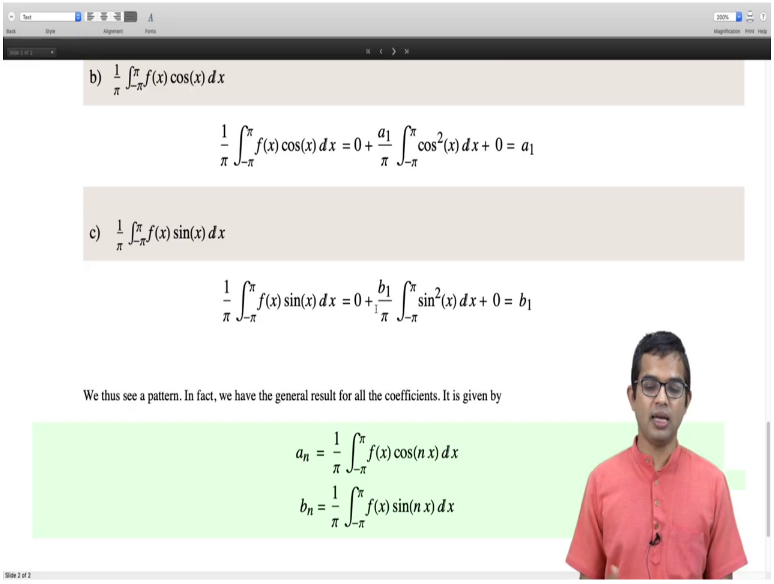

And then you have integral minus pi to pi f of x, if you tag along on the cosine of f x dx and then you do this integral, then you can quickly show your show that every term except the you know this term involving cosine, you know cosine x you know goes along with cosine x and it leave. Whereas because of this result that we have had, right, they are all orthogonal to each other, every other term will vanish and so, you get a 1, right. You can check that this integral will give you a 1.

And likewise we can check that in fact, if you do this integral f of x with sine of x and multiply with sine of x and then do the integral, every term except the one with sine of x will vanish because of this property of orthogonality and this will give you just b 1, right. So, this is a pattern which you should carefully check.

And in fact, this will extend which also is something that you should directly work out, and what we see is if you want the nth coefficient a n you just simply have to multiply f of x with cosine of n x. And what it does is because of and then integrate in the period minus pi to pi. And what it will do is it will annihilate or it will destroy every other term and it will only selectively choose the term that you are interested in and work out for you the coefficient, right.

So, in some sense this is just taking an inner product of your vector with the basis vector of your choice, right. That is what is going on, right. So, it helps to think of this from the point of view of you know vector spaces and so, you get a n and b n. So, now, you see that you have these compact expressions for a n and b n, right. That is why we wrote a n a naught by 2 instead of just a naught for the constant term, right. So, that we have this one nice formula it is easy to remember, right.

So, what we have managed to show in this lecture is that you know Fourier series expansion of a periodic function with period 2 pi, in terms of cosines and sines, you know is there we have provided a prescription for this, we have worked out the coefficients and we have also argued that.

It is really, the Fourier series expansion is actually an expansion of a vector in terms of an orthonormal basis, right, so where sines and cosines should be thought of as providing a basis for this vector space, ok. That is all for this lecture.

Thank you.