Mathematical Methods 1
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Fourier Series
Lecture – 44
Piecewise continuous functions

So, we have started our discussion of periodic functions and our goal is to be able to expand you know periodic functions in a Fourier Series right. So, before we give out the prescription for doing this, it is useful to look at functions and continuity, you know look at it closely and also, bring in a you know linear vector space perspective you know for this functions right which is the objective of this lecture ok.

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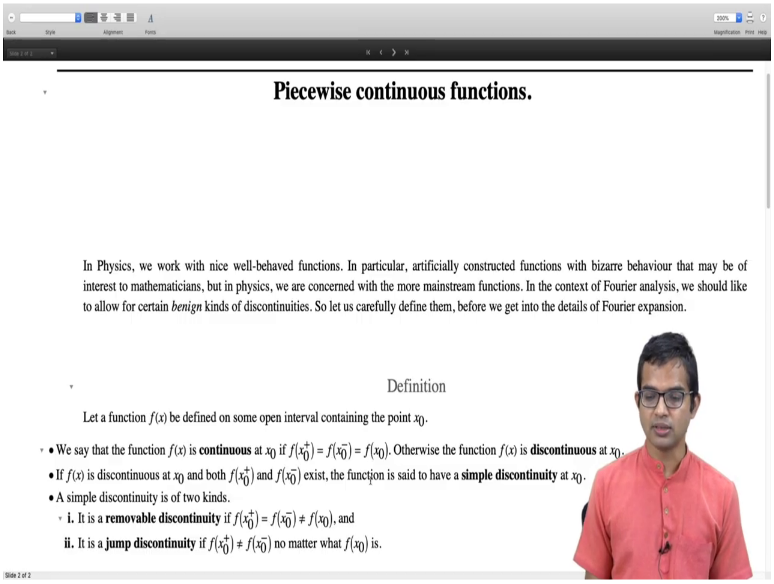

So, in physics, we will typically work with you know nice well-behaved functions. And now, you know in particular, we do not worry too much about you know artificially constructed functions which have very strange properties right.

Although these are important, when you are building a general theory, mathematicians are more picky. Whenever you have a broad statement or a broad structure that you have built, they try to come up with examples which can break apart the structure. So, that you know they worry a lot about how things can be stated very precisely right.

And so, a lot of work goes into making things extremely tight, which is not the approach that we will take. By and large from a physicist perspective, you know we like to work with smooth functions right.

Rather to be the word smooth itself it is being you know indicate more carefully what it means right. So, let us define two terms. So, let us think of a function which is defined on some open internal and you know containing a certain point x naught. We say that this function is continuous right; intuitively, we know what continuity is.

So, the key point is that you know the value of this function not only at x naught is known; but the limit of you know this function slightly to the right of this is the point of interest is also well-defined and not only is it well-defined, its value slightly to the right of where you are is the same as you know the value of this function slightly to the left right that is the notion of continuity right.

So, a function which is not continuous at a point is said to be discontinuous is at such a point right. So, not all discontinuous functions are discontinuous in a violent way. There is a relatively benign kind of discontinuity which we would like to allow. Because these are also the kind of functions which appear all the time in physics right.

So, these are the functions which you know which have what are called simple discontinuities right. So, you say that a function has a simple discontinuity at a point x naught, if the limit f of x naught plus exists and the limit f of x naught minus exists right.

So, if it were a violent kind, then one or both of these elements would not exist and then, it would require a you know different kind of treatments, such functions which more violent kinds of discontinuity right. In among functions with simple discontinuities right, there again you know there is further classification. The simple discontinuity itself is of two kinds right; one of them is called a removable discontinuity right.

So, you know everything is good with this function except that you know you have defined the value, you have given the wrong value for this function at just one point right; everything else otherwise is just nice and continuous about this function. So, the only problem is that the limit f of x naught plus will not exist, limit of f x naught minus also exists and they are equal, but only the value of this function is some other values and you know these two right.

There is so it is called a removable discontinuity because its removable right. You could just redefine this function by shifting the value to the value of these two elements and then, their discontinuity is gone right, such a discontinuity it is called a removable discontinuity right.

And then, there is another kind which is also simple discontinuity which is called jump discontinuity right. So, if f of x naught plus and f of f x naught minus they exist, but they are not the same. And it does not matter what value you have prescribed to f of x naught.

There is no simple way to fix this discontinuity it is a discontinuity, but it is something that you can work with right; f of x naught plus exists and f of x naught minus also exist, they are not equal and f of x naught, you can choose to be some value right that lies in between that oftentimes one chooses it to be you know the average of f of x naught plus and f of x naught minus right. We will come across this later.

So, the key point is that you know if a discontinuity is simple, then you know we can work with this right and these kinds of discontinuities appear all the time and we want to include this in our discussion right. So, that is why we are making these definitions right away.

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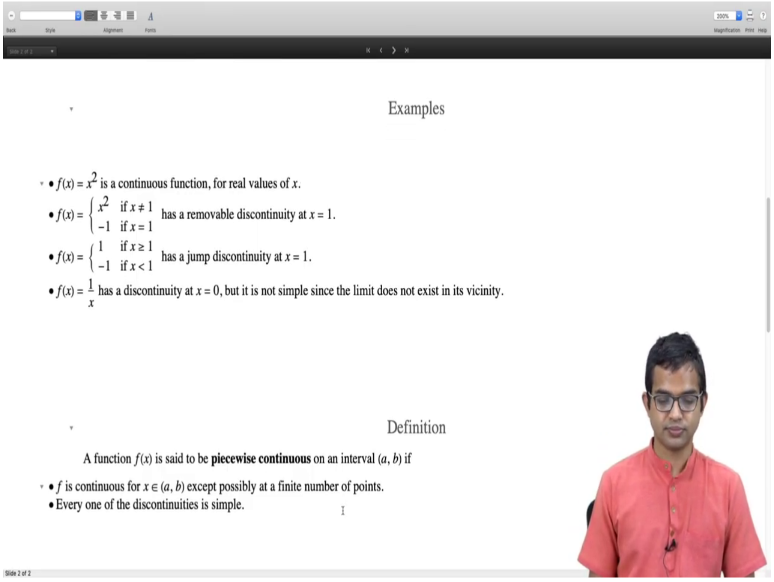

And so, let us look at a few examples. So, if you have a functional f of x equal to x square, then it is clearly a continuous function for all real values of x. No problems. But if you say f of x is x squared if x is not equal to 1; but at x equal to 1, you say that this function is minus 1 right.

So, clearly there is only a problem at x equal to 1 because according to you know this function, it should be 1 you know. So, this is what would be a removable discontinuity right. It can be removed, if you simply redefine this function to be 1 at x equal to 1 and then, you do not need to make this distinction between x naught equal to 1 and x equal to 1. It is something that you have healed.

On the other hand, if you have a function which is you know plus 1, if x is greater than or equal to 1 and if it is minus 1, if x is less than 1, this is a jump discontinuity right. So, this is like a step function and so, here you have defined the value of this function to be equal to the right limit at x equal to 1 right. So, but it does not matter, it is a jump discontinuity right.

So, I have also given you one example of a more violent kind of a discontinuity. 1 over x, it has a discontinuity at x equal to 0; but it is not simple because the limit does not exist in its vicinity right. So, with a simple discontinuity, we will be able to work with these functions even in the context of Fourier series right. So, that is what is coming ahead.

So, let us look at a few more definitions. So, a function is said to be piecewise continuous on an interval a comma b, if f is continuous you know in this entire interval except at a finite number of points right. It has a bunch of discontinuities, but the number of discontinuities is finite.

You cannot have an arbitrarily large number of discontinuities, if there is only a finite number of them and each of these discontinuities is simple right. Then, you have a piecewise continuous function. These are also kind functions which we can work with ok.

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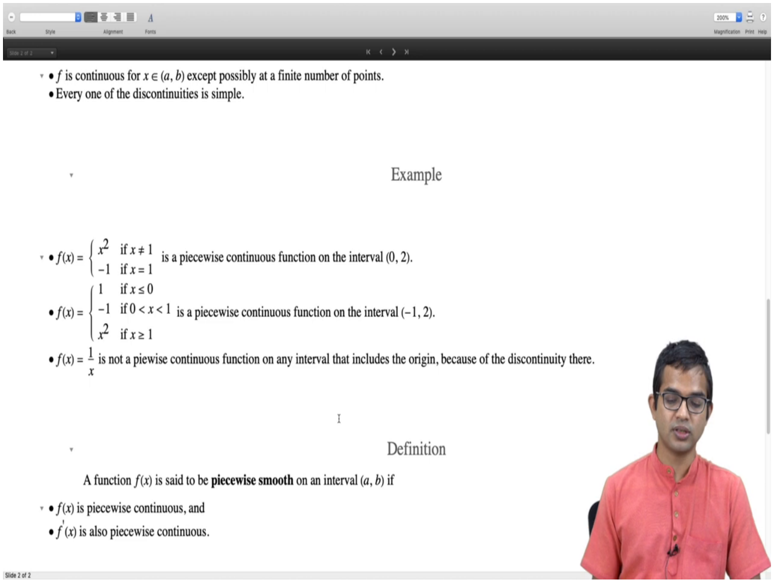

Let us look at an example; a few examples. So, you if x f of x equals. So, it is the same examples, we look at them again; now that we have this notion of piecewise continuous. So, if you have f of x equal to x squared x naught equal to 1 and equal is minus 1, if f x equal to 1.

It is piecewise continuous, and the only difficulty is at x equal to 1. So, it is piecewise continuous in this interval 0 comma 2. On the other hand, you have if you have minus 1 and x squared. Second example, it is piecewise continuous again, which is there is the jump discontinuity, but it is piecewise continuous right.

So, if there is a finite number of discontinuities, it is not a problem and here it is there is just 1 discontinuity and that is a jump discontinuity. But on the other hand, f of x equal to 1 over x is not going to be piecewise continuous. Provided here you are looking at an interval that includes the origin right.

Because there is a mess at the origin and that is not an easy kind of mess to deal with right. So, these are the type of discontinuities which you know which we would like to avoid right in you know you know you can come up with complicated functions you know something like sin of 1 over x right you can think of functions of this kind right.

So, you can come up with weird functions like you know if f of x is a certain value, if x is rational and if x is irrational, there is some other value and so on right. So, these are the kind of constructions which would be of more interest for a mathematician. So, we will keep such functions aside and look at you know these smooth functions right.

When you talk of smooth, let me define the word smooth also a little more carefully here. So, not only is a function which is piecewise continuous is interesting; but in fact, there are functions which are piecewise smooth that are even more interesting, hm right.

These functions not only are piecewise continuous. So, but they are also you know their derivatives are also piecewise continuous and so, these are called piecewise smooth functions on some internal a comma b. A function is piecewise smooth if it is piecewise continuous and its derivative is also piecewise continuous right.

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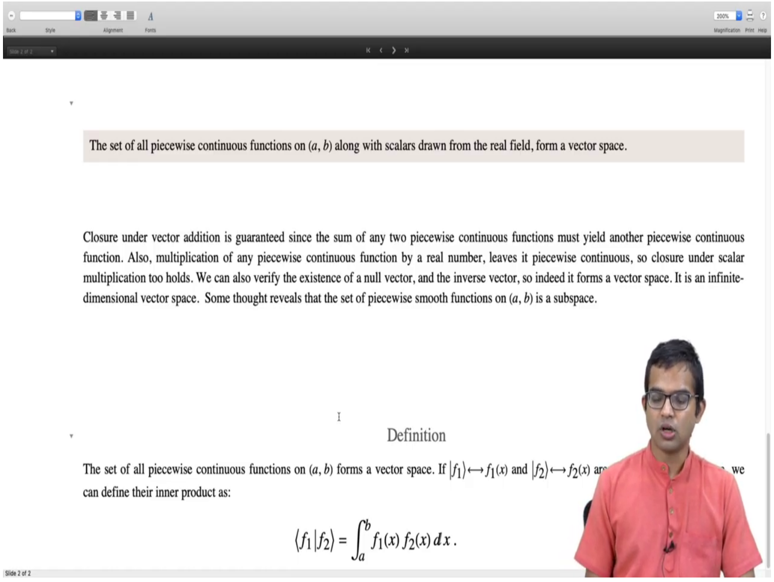

So, it turns out that the set of all piecewise continuous functions on a comma b, you know along with scalars drawn from the real field, form a vector space right. So, it is something that we can verify right. What do we have to verify? If you take any two such vectors from this box of vectors and add them, you should be able to get another vector which also lies in the same box right. So, which is true right.

Closure under vector addition is true because the sum of any two piecewise continuous functions is also going to be piecewise continuous. If you take any piecewise continuous function and multiply by a real number, you will get another function which is also piecewise continuous.

So, closure under scalar multiplication too holds and indeed, there is a null vector which is just the function f of x equal to 0. There is an inverse vector also, you can check for any function and such that if you add the inverse to this vector, you will get back the 0 vector.

So, an important observation to make is that this is an infinite dimensional vector space and some thought also reveals that in fact, the set of piecewise smooth functions is a subset of this and in fact, it is a subspace because again you can verify closure under vector addition. If you take two piecewise smooth functions and add them you will get another piecewise smooth function right, closure under vector addition and scalar multiplication and so, it forms a vector space.

So, it forms a vector space and in fact, we can also define an inner product for this vector space right. So, if you think of a vector f 1, a function f 1 of x and f 2 is another piecewise continuous function. Then, we can define the inner product as you know inner product of infinite two is the same integral a to b f 1 of x times f 2 of x d x right. Now, I am just thinking of all these are real valued functions and all those scalars are real right.

It is possible to also come up with a you know somewhat more you know to extend this definition, when you are working with you know complex scalars and then, you will have to introduce a complex conjugation and all. So, but for now, let us just say that you are working with a real field and so, we have this inner product defined like this and you can verify that this is a valid inner product right.

So, why do we bother about thinking about this as a vector space and you know thinking of all this - the notion of an inner product and so on right? So, we will see ahead that it's useful to bring in our linear algebra understanding and you know when we develop this prescription for Fourier series, we will also use our insights from linear algebra. So, we have a deeper understanding of what is going on right.

So, there are two aspects; one is the operational aspect: given a function, we would like to be able to work out the Fourier series. But also, since we already have this nice training in linear algebra, it's useful to see that there is this underlying you know algebraic structure to all of this right. So, that is the spirit in which we have introduced this lecture. In the next lecture, we will look at the prescription for working out a Fourier series of a periodic function. That is all for this lecture.

Thank you.