Mathematical Methods 1
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Fourier Series
Lecture – 43
Average value of a function

So, we have motivated how it may be possible to represent a periodic function as a series of, as a sum of a bunch of sines and cosines right. So, and specifically, we looked at the example of the harmonics and overtones of a musical note right.

So, that is the problem of Fourier Series, but before we go there, we will before we describe you know like a prescription for how to get the Fourier series for a periodic function. Let us look at the idea of the average of a function - this is the subject for this lecture.

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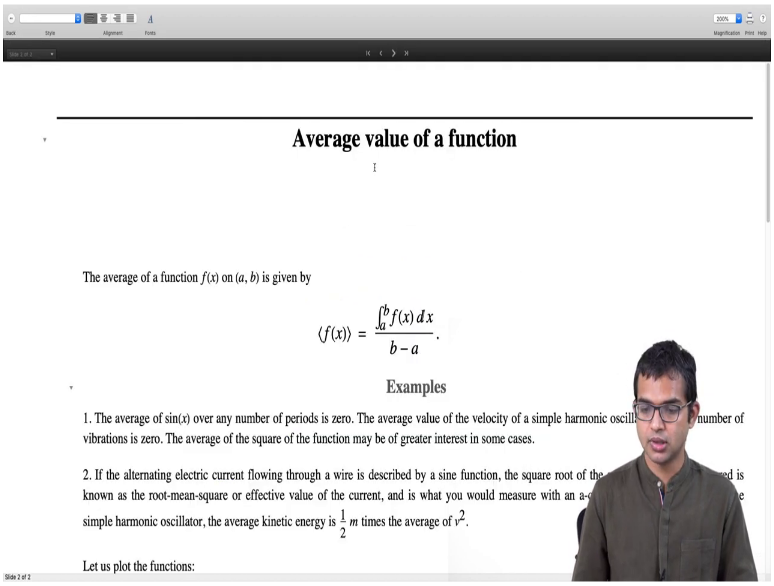

So, the average of a function is something that we know we can understand intuitively. Given a bunch of numbers we would just add them up and take the sum and divide them by the number of you know these numbers that you are working with.

And likewise, if you are given a function on some interval right, you just integrate, which is like a sum itself. There is a limiting procedure that we are considering and then, you divide by the length of the interval right. So, this is the notion of the average of a function f of x on an interval.

Now, you know the average of a function appears in many contexts right. So, let us look at a few examples. So, the average of sin of x over any number of periods is going to be 0 right as long as you take a full period. If you take an average over half a period, then it is not going to be 0 right.

So, the average value of the velocity of a simple harmonic oscillator over any number of vibrations is 0 right because it is really the idea of you know averaging over a sinusoidal function which is going to give you 0.

But many times, you know more information may be contained in the average of the square of this function right. So, you know if a function is spending equal amount of time you know above the x axis as it spends or the time axis, above the time axis as it spends below the time axis, then you know although on average, the function itself may be 0 it is still you know the square of this function may also contain information right.

So, many times, it is useful to consider the average of the square of your function right. For example, when you are looking at alternating electric current right so, we know that it's a periodic function and the square root of the average of the sine squared is known as the root mean square or effective value of the current right. If you just blindly do an average of this current function, you would just get 0 right, but clearly the current being 0 is not an accurate way of representing what it is doing right.

So, we know that the current can definitely do things for you which means that it is not just a 0. So, here a more useful quantity to you know to measure the current is in fact the root mean square or effective values of current and which is what you would measure if you took an a-c ammeter right.

So, in the example of the simple harmonic oscillators, the average kinetic energy is something that you might be interested in measuring right so, that would be given by half m you know average of v squared right.

So, when you are looking at the kinetic theory of gases, you know you have lots of particles and there too you are interested in measuring the average of you know v squared is of great importance right. So, you might have seen these concepts in the context of the equipartition theorem in statistical mechanics; these kinds of averages are of importance.

Sometimes one is interested in the time average, sometimes one is interested in some ensemble average and so on right. So, all kinds of averages are of interest right. So, for our discussion, we are looking at you know the properties of just a one-dimensional function right and so, the average is taken on certain intervals here right.

So, you know the most basic thing that you can do given an entire function and you want to bring it down to just one number is to just take its average - a crude representation of the information contained in the entire function is just this one number.

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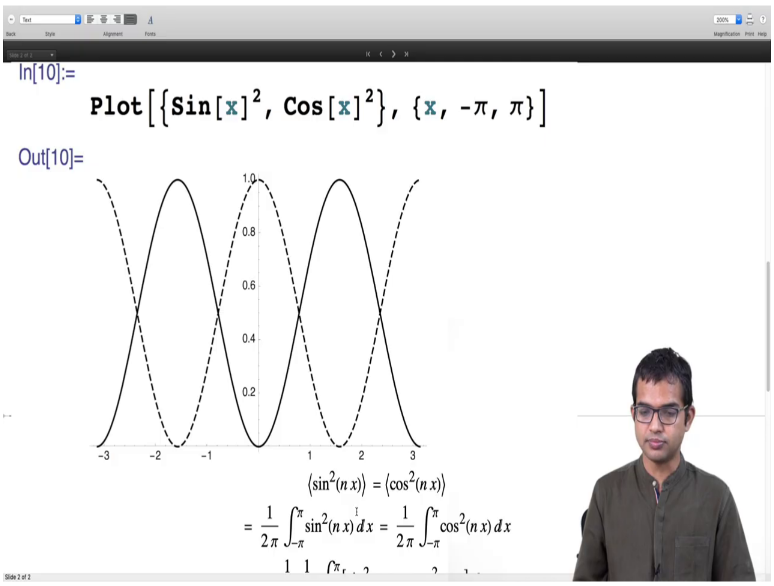

So, for the purpose of this discussion, let us say that you are looking at the average of sin squared n x and cosine squared of n x in from in the interval minus pi by minus pi to plus pi. So, let me fix the interval to be minus pi to pi by now and then, you will see in a moment where we can actually shift this period about right, we have much more freedom.

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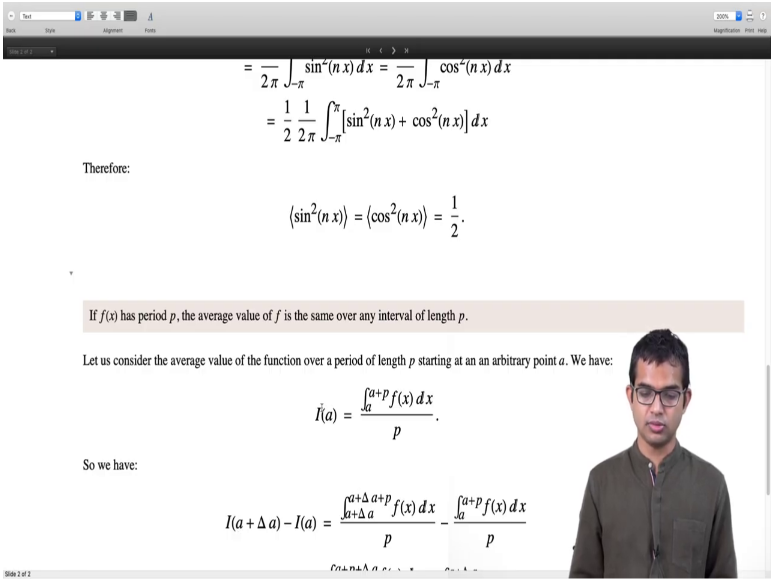

So, if I do this, then I will see that in fact, since this quantity is equal to this quantity, I can just add the 2 and then divide by 2. So, it is half of 1 1 over 2 pi integral minus pi to plus pi sine squared of n x plus cosine squared of n x d x. But this quantity is nothing, but 1 right.

And so, therefore, I am going to get just a half. So, the average value of sine squared of n x right is half right as long as you have taken the average over a multiple of the period of this function right ok. So, let me give you a more general argument. So, if you have a function which has some period p right. So, the average value of f is the same over any interval of length p right.

So, I have looked at you know I have taken it from minus pi to pi to be safe in this argument right, but you all you need to do is find the period of this function whichever function it is right if you have an f of x, an arbitrary function of by a periodic function with period p so, the average value of this function f is the same, no matter which interval of length p you consider right. So, the argument is the following.

So, let us consider the average of this function over a period of length p starting at some arbitrary point. So, I am interested in this integral, integral a to a plus p f of x d x and then, I have to divide by the length of the interval which is going to be p right. So, I of a. So, you would think that this is you know a priori you might expect this to be dependent on a.

But let us look at what happens if I you know take you know compute this at some other point a plus delta a right. So, this delta a does not have to be infinitesimal, I am not even thinking of that at this point, I am just considering some other point which is at a distance delta a away from a.

So, I of a plus delta a minus I of a is you know i am just in place of a, I have to put a plus delta a so, I have a plus delta a here, I have a plus data a plus p f of x d x minus this original integral.

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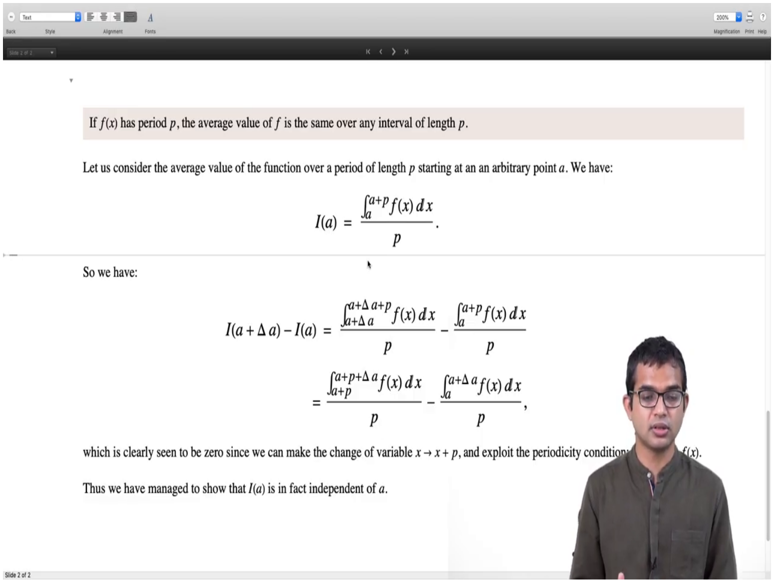

So, then I notice that there is this; there is this interval which is common to both of these integrals and that is basically a plus delta a all the way up to a plus p right. So, there is a leftover bit which is from a plus p to a plus p plus delta a so, that I have to keep and then, I have to also subtract this other part which is also left out which is a to a plus delta a. So, that part is also left out right. So, from here you see this.

And therefore, I have you know these two small small components or big components does not matter, it is something that we cannot say because delta a is arbitrary right. So, I have, but it is precisely this component and this component remains, but now, we can argue that.

In fact, these two are exactly the same right, these two will cancel and you will get 0 and the reason is you can just do a change of variable x can be made to go to x plus p and then, you will immediately see that if you invoke the periodicity of this function f of x plus p is equal to f of x, therefore, both these integrals are the same and the denominator is the p is the same p in both. Therefore, the difference of this will be 0.

Therefore, we have managed to show these are I of a plus delta a is equal to I of a. So, therefore, I of a is actually independent of a, you could have started at any point as long as you cover a length p which is a period of this function you are going to get the same average value right.

So, we will see in our discussion which you know which will work out the Fourier series representation for this for an arbitrary periodic function in fact, the average of this function is like one of the basic ingredients which goes into this Fourier series.

And then, there are more complicated, there are other elements which we will cover which will come up right in our discussion, but this lecture is simply warm up exercise once again, but you know we are ahead if we have taken one more step towards working out the Fourier series of a periodic function which is coming ahead. That is all for this lecture.

Thank you.