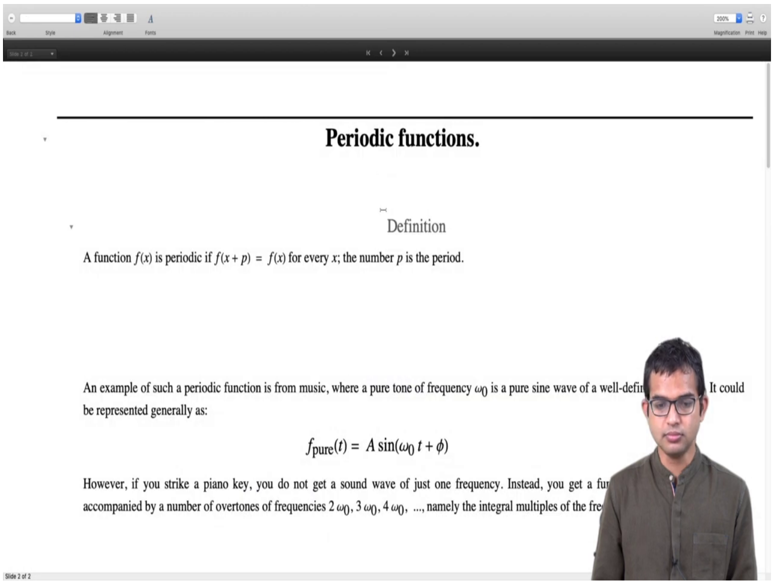
Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Fourier Series
Lecture – 42
Periodic functions

Ok. So, in this lecture we will look at periodic functions, what they are and some of their elementary properties. Periodic functions are a basic idea or basic ingredient you know in our discussion of Fourier series which is coming up a little bit later, but in this lecture we will look at some basic properties of periodic functions ok.

(Refer Slide Time: 00:47)



So, we start with this definition. So, a function f of x is periodic if f of x plus p is equal to f of x for every x, right. So, inherent in this definition is also the idea of p which is the period. So, the number p is called the period right. So, immediately from this definition it is very intuitive right. So, we have come across periodic functions. And, again you know closely related to the idea of in the context of simple harmonic motion for example, right.

So, you have seen periodic functions and they are ubiquitous in physics. We observe a number of phenomena which are periodic in nature. We also work with systems where you know you have an external force or a drive which you can tune.

And, how this system behaves under the application of such forces is very important; these ideas appear in all kinds of fields and subfields of physics. So, periodic functions are of great importance in physics.

Now, a periodic function is like what we have defined and the number p is the period right. So, immediately from this definition we see that if p is a period then, so is any multiple of p also must be a period right.

So, you know typically we are interested in you know the smallest such p right, that is what we mean by the period; although strictly by speaking from this definition you know p if p is a period, then so is 2 p right because f of x plus 2 p is equal to f of x plus p which is equal to you know f of x right.

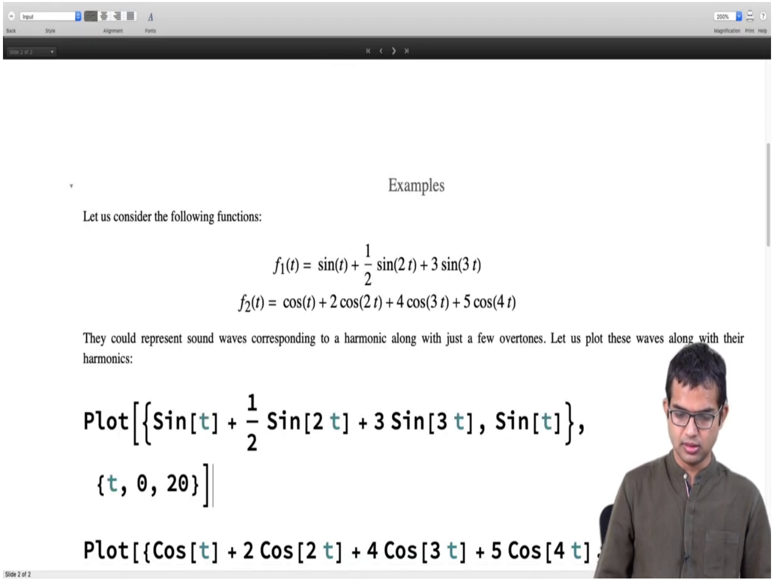
So, we are usually going to be interested in the smallest such p and that is what we term as the period. So, an example of a periodic function you know we can think of from music player if you are producing a pure tone right some instruments you know something like a tuning fork or something which can produce a pure tone of frequency omega naught and it is perfectly you know sinusoidal tone and it has just one well defined frequency.

And, then you could represent that with some sinusoidal function. So, it is just like in simple harmonic motion you have A times sin of omega naught t plus phi, but in general you know even very elementary you know instruments right. So, produce much more complicated sounds, right.

So, if you strike a piano key you might think that you are just you know playing as just one pure note, but in fact, this pure note is going to contain not just one frequency, but it is going to have you know what is called a fundamental and all multiples of this fundamental frequency which are called overtones right.

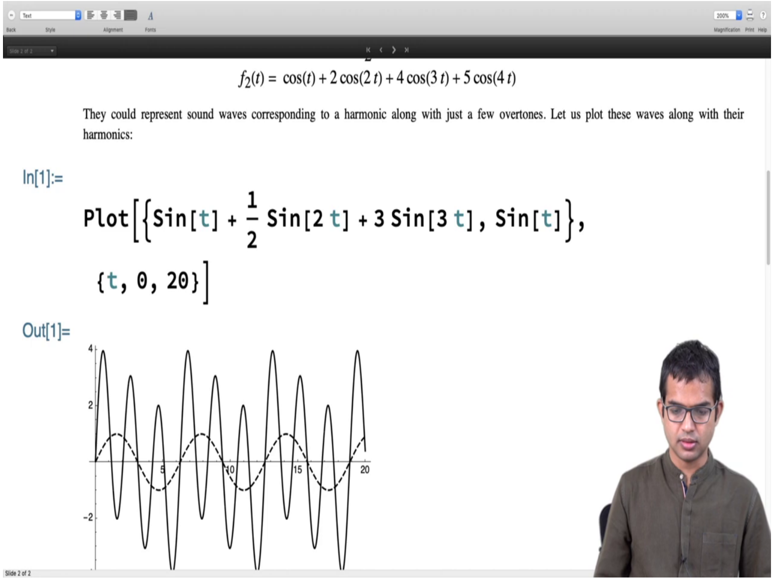
So, if the fundamental has some frequency omega naught, you would expect to hear you know superpositions which are of frequencies 2 omega naught, 3 omega naught, 4 omega naught so on and all integral multiples of the frequency of the fundamental are going to be present, right.

(Refer Slide Time: 04:14)



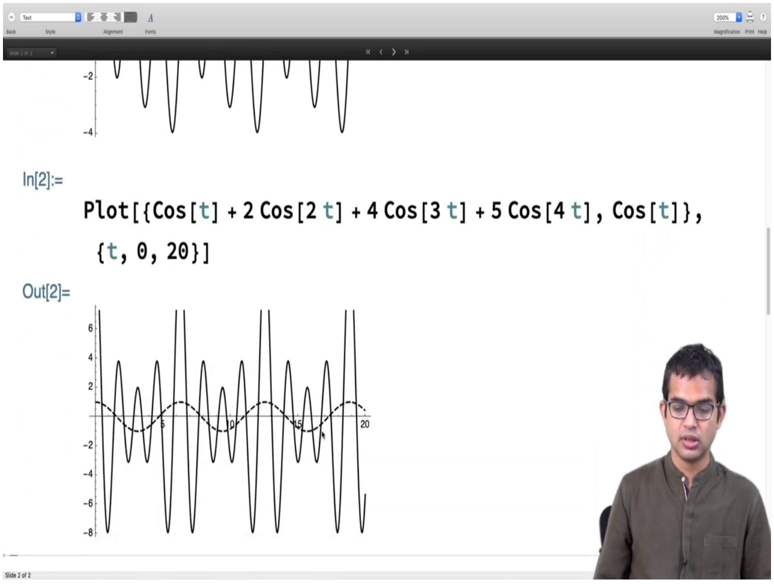
So, suppose you consider these two kinds of functions. So, here I have f 1 of t is sin of t plus half sin 2 t plus 3 sin of 3 t right. So, which are just some you can consider some arbitrary coefficients right I am just considering you know a harmonic here. And you know these 2 overtones here and a harmonic here and 3 overtones here, right you can play with this you can consider even a larger number a smaller number of overtones and so on.

(Refer Slide Time: 04:50)



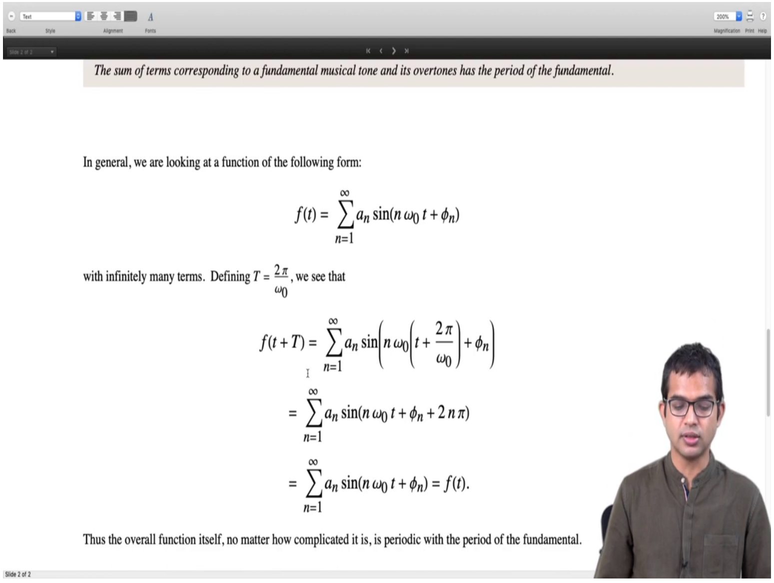
So, I want to plot these two functions and yeah. So, you see if I here I have plotted you know this function f 1 of t and then also I have plotted just the fundamental right. So, what I want to observe from this plot.

(Refer Slide Time: 05:08)



And, I have one more plot which is again for the second function and you know in comparison with its fundamental frequency. So, I see that this very complicated function which comes from just the superposition or just 3 or 4 harmonics already looks quite complicated, but you see that if you observe carefully that you know they had there is this repeating tendency. And in fact, it is going to repeat exactly like the fundamental does, right.

(Refer Slide Time: 05:43)



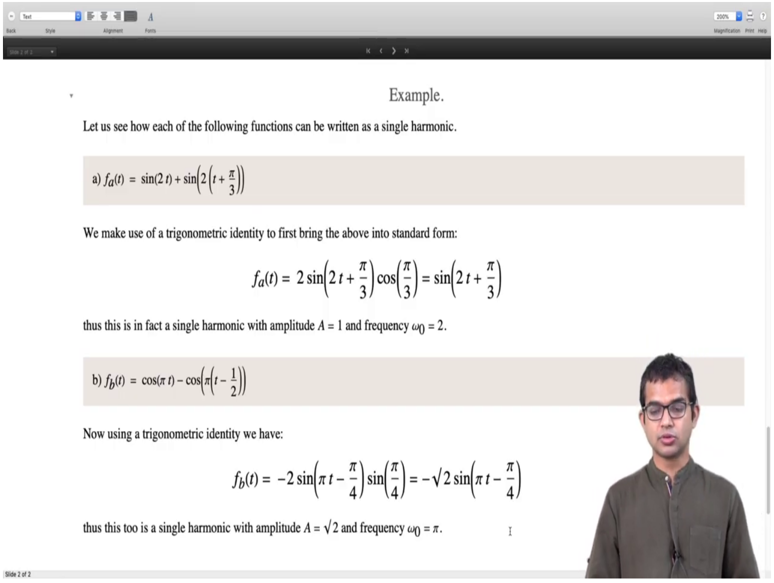
So, what you can say is in fact, the sum of terms corresponding to a fundamental musical tone and its overtones has the same period as the fundamental right which we can explicitly verify, right. So, if you have a function f of t which is made up of you know sin omega naught t plus phi n plus sin of 2 omega naught t plus phi 1 and phi 2 and so on, right.

So, I have a bunch of coefficients which in principle can run all the way up to infinity. So, it can you can have an infinite series of this kind right. So, the point is that term by term every term has exactly the same period right which we can explicitly verify, right.

So, defining t is equal to 2 pi by omega naught we see that f of t plus T is equal to summation over you know wherever we have t we have to put t plus 2 pi by omega naught and then if you expand this you see that there is a this you know is the same as writing sin of n omega naught t plus phi n plus 2n pi and we know that you know the sin function is a periodic function.

So, if you add an integral multiple of 2 pi to its argument. It remains unchanged. So, in fact, term by term this function is unchanged when you increment t by capital T. Therefore, f of t plus T itself is equal to f of t. So, in fact, the overall function itself has exactly the same period. No matter how complicated the overall function looks it is going to be periodic and with the same period as the fundamental, right.

(Refer Slide Time: 07:25)



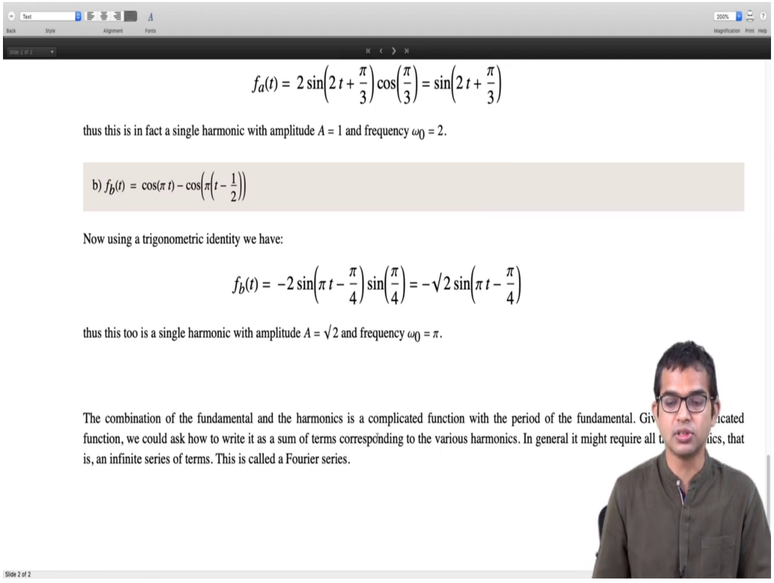
So, I also want to consider another example where, just like we saw how functions which look very complicated apparently, but in fact, they referred to simple harmonic motion. So, likewise here you could have apparently very complicated looking functions, but actually they refer to just the single harmonic.

So, like in this example sin of 2 t plus sin of 2 t plus pi by 3, but if I invoke the standard trigonometric identity involving sin c plus sin d then in fact, I see that f a of t is the same as sin of 2 t plus pi by 3.

So, although it looks like there are two terms right if you have to arrange them properly and then you can bring it into this for standard form. And then you see that in fact, it is just a single harmonic with you consisting of just the fundamental harmonic with amplitude A equal to 1 and frequency omega naught equal to 2.

And now this is an example where again invoking the trigonometric identity you can verify that this is just minus square root 2 sin of pi t minus pi by 4. Therefore, this too is just a fundamental harmonic with amplitude square root 2 amplitude like we said is necessarily positive and frequency omega naught is pi, right. So, this is just to illustrate that you know one must be careful with looking at functions and bring them into standard form before drawing conclusions.

(Refer Slide Time: 09:05)



So, the main moral of this discussion is that, if you have a fundamental and you know superpose it with a bunch of harmonics, then the resulting function can be extremely complicated, but still it has you know the same period as that of the fundamental.

And oftentimes we are interested in the opposite question which is that you have given a signal which is incredibly complicated right and we want to ask if it is possible to think of this very complicated periodic function in terms of various harmonics, right. So, this is basically the heart of the Fourier series problem.

And, then when you are able to express a complicated periodic function in terms of you know a fundamental and its harmonics you know various coefficients if you are able to pull them out. Then, you can just work with all these coefficients and understand what is the contribution of various harmonics or overtones to this problem. And it gives us a lot of insight if we are able to do this.

So, as we will see what is the prescription for working this out; and then we will look at how general this method is and how it can be used. All of these are coming ahead. That is all for this lecture.

Thank you.