Mathematical Methods 1
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Linear Algebra
Lecture – 40
Function of matrices

Ok. So, this is going to be the final lecture on Linear Algebra. So, before we move on to the next topic, in this lecture, we will look at you know you know this one advantage of diagonalization. So, we talked about diagonalization of matrices. We talked about diagonalizability, so why bother with doing this right.

So, among other things, it can help us work with Functions of matrices that is the subject of this lecture and then, we will also look at a theorem which is an important theorem which is, you know, which can be profitably discussed in this context.

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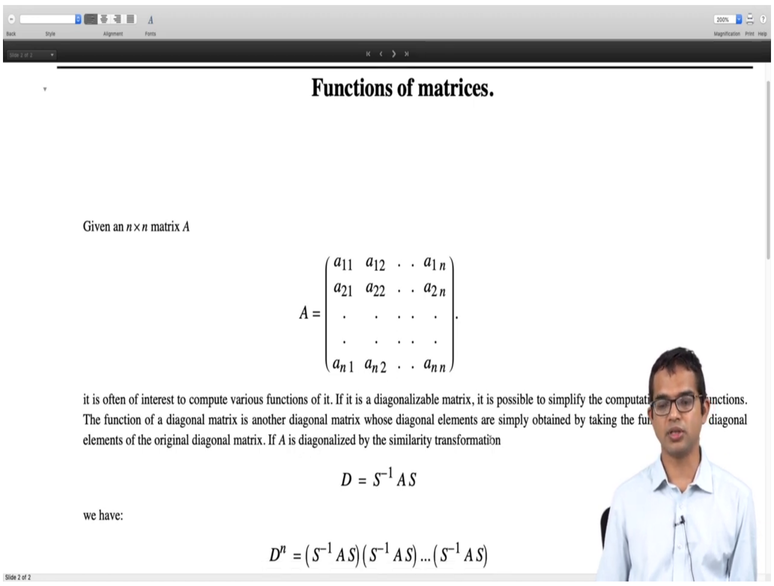

So, given an n by n matrix A right. So, you know oftentimes, we are interested in looking at what A squared is that is going to be another matrix, a cube is you know A power n or in general some function of this matrix A. Now, if it is a diagonalizable matrix, then you know this kind of a computation is greatly simplified.

So, all you have to do is work out the similarity transformation which diagonalize it if you can somehow able to work this out D is equal to S inverse A is AS, then if you want to take the nth power of A, you just take the nth power of D which is much easier to do because just taking the powers of the eigenvalues D to the n is S inverse AS times S inverse AS all the way up to S inverse AS n times, then you see that S and S inverse they collapse with each other and you get identities everywhere.

So, then you have A times, A all the way up to you know the nth A and you have an S so, which is you can group together, and you immediately see that this is the same as S inverse A to the n S right.

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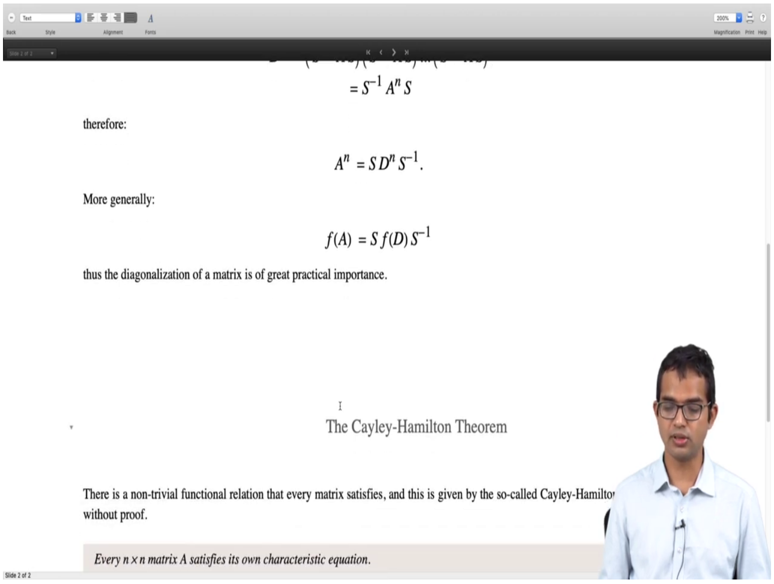

Now, we can multiply the left-hand side and right-hand side with S on the left-side and S inverse on the right-hand side so, and then, we rearrange terms and you have A to the n is the same as S times D to the n S inverse.

So, if you can find the similarity transformation S which can you know diagonalize your matrix and of course, you know the eigenvalues, then it is very easy to compute A to the n right and in general, you can find some function of A using this the diagonal matrix and similarity transformation right.

So, this has you know great practical importance. So, diagonalization of matrices is important and there are you know for big matrices, it is a whole industry to find to diagonalize matrices and there are very very sophisticated algorithms available for carrying this task out right, but we are not going there, but I am telling you that lots of physics problems involve diagonalization.

Sometimes diagonalization of large matrices and you know many calculations which involve matrices are best handled with the help of you know diagonalization routines right. So, this is just a very very brief discussion of you know how diagonalization can help us to compute functions.

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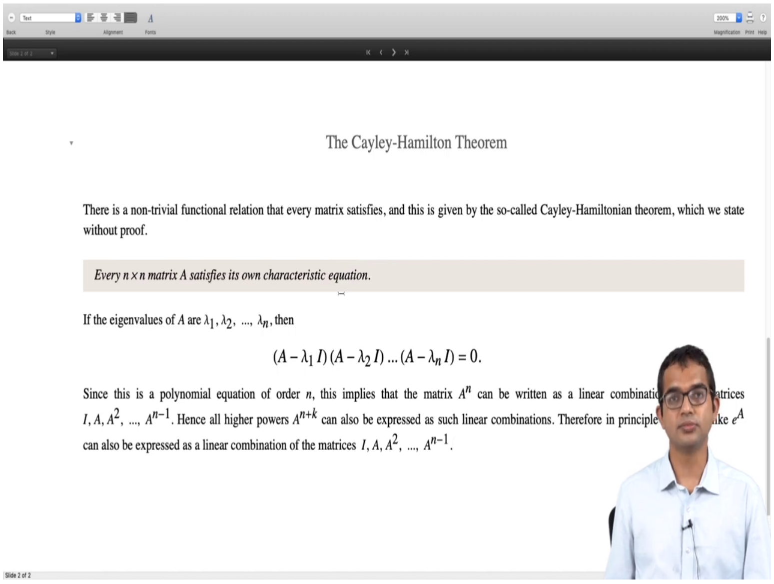

So, now let us look at a special kind of a function which is associated with any matrix and that leads us to this theorem called the Cayley-Hamiltonian theorem which is a non-trivial theorem which says basically that every matrix satisfies its own characteristic equation right. So, very simple to state and fairly easy to understand, but it is a non-trivial theorem. Of course, we are not going to prove it right.

So, what is the characteristic equation? So, the determinant of A minus lambda I is equal to 0 right. So, the characteristic polynomial you can also think of it as A minus lambda 1 times I into A minus lambda 2 I all the way up to A minus lambda n I if you know all eigenvalues right; so, in fact, to use this characteristic equation to compute the eigenvalues, you start with the determinant of A minus lambda I equals 0, then you work out lambdas the roots of this polynomial equation right.

And then, what Cayley-Hamiltonian theorem says is in fact, the matrix itself satisfies the Cayley, the characteristic equation. So, once you have got all these lambdas. So, notice that this is a statement about the eigenvalues right; at this point we have not said anything about eigenvectors.

So, this is true for any n by n matrix. You find all eigenvalues, you can put your n by n matrix into the characteristic equation because it always exists for any n by n matrix defective or not does not matter right, this is not about eigenvectors and so, you this equation is satisfied you know by any matrix correspond with the lambdas are the eigenvalues of this matrix.

Now, this is a polynomial equation of order n so, this means that you know you can expand it out and it is like A to the n, some coefficient times A to the n plus some coefficient times A to the n minus 1 plus some coefficient times A to the n minus 2 so on all the way up to down to I is equal to some coefficient times I is equal to 0.

Now, which means that you will be able to write A to the n as a linear combination of the matrices I, A, A squared all the way up to A n minus 1. So, in general, any higher power A to the n plus k can also be written as linear combinations of these matrices. So, in fact, therefore, you can write any function of your matrix f of A itself as you know linear combinations of just these n minus 1 to matrices I, A, A squared all the way up to A to the n minus 1.

So, in fact, in principle a function like e to the A can also be expressed as a linear combination of these matrices right. So, I have just stated this as a fact and it is a useful fact sometimes, but from the you know point of view of completeness, we include just a short description of the Cayley-Hamiltonian theorem right.

With this, we come to an end you know for our discussion or for Fourier into linear algebra, it is a vast subject and if you want to carefully get into all the details, it is a you know like a full course in itself, but I hope hopefully we have covered all the salient aspects of the very beautiful and you know rich theory of linear algebra.

And we will move on to the next topic starting from the next lecture, but some of the results you know drawn from linear algebra will be used later on also in other topics that we cover ok. That is all for this lecture.

Thank you.