Mathematical Methods 1
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Linear Algebra
Lecture - 04
Dot product of Euclidean vectors

So, we have been discussing linear algebra. So, we saw how it is possible to go from you know vectors and define a more abstract vector space. So, before we go further we will go back to the usual kind of vectors, you know maybe we will adopt this strategy from time to time whenever we want to introduce a new concept or when possible, we will step back, and look at familiar vectors you know look at a concept from there and see how you know that particular concept can be generalized. So, today in this lecture we are going to look at you know the notion of a Dot product ok.

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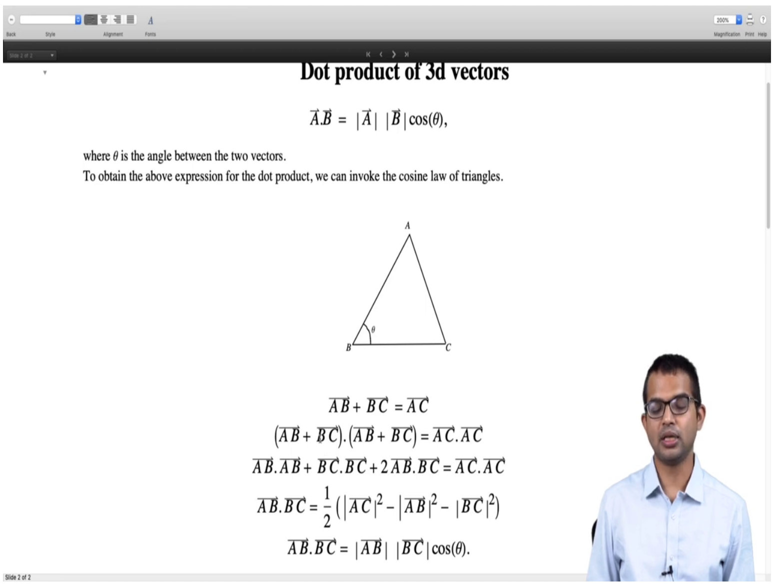

So, the dot product we are all familiar with is just defined as you know like in this notation here A dot B is just the magnitude of A times magnitude of B times cosine of the angle between them right for 3D vectors, this is how we are used to thinking about it right. So, one way to see this is actually to invoke the cosine law right. So, I mean, so you have to consider some triangle ABC, so we know that the sides AB, BC and CA can be related with the angle theta right, one of the angles I am taking the angle ABC to be theta right.

So, we know that the vector AB plus vector BC is equal to vector AC right. So, now, if you take this the dot product of the left hand side with itself and the right hand side with itself, so you have AB plus BC dotted with AB plus BC is equal to AC dotted with AC. And, then we expand and so we have AB dot AB plus BC dot BC and plus BC dot AB plus AB dot BC, both of them are understood to be the same, so that plus 2 times AB dot BC is equal to AC dot AC.

And then we push all these terms involving you know just the magnitude squares to the right hand side, so then we are just left with AB dot BC right. A is equal to half of you know this the modulus squared of you know one side minus the modulus squared on another side minus the modulus squared of the third side right.

So, we identify the left hand side as (Refer Time: 02:53) quantity that we are interested in the dot product between two vectors is just the magnitude of ah you know each of these sides times cosine theta which come from the cosine law of triangles right.

So, one could either if one already accepted this expression as the dot product, one could use this as a way to get to the cosine law, but we have an independent way of getting to the cosine law. So, you know this is like an argument for why the dot product of two vectors is given by this expression right.

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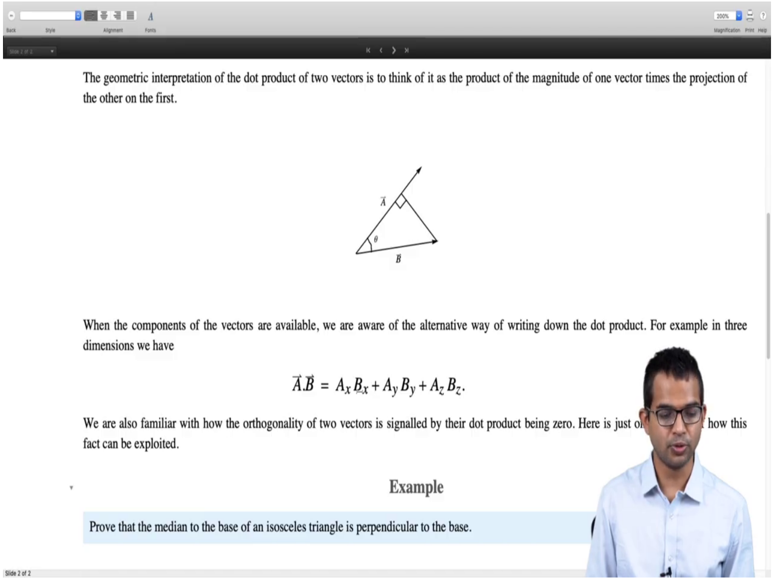

So, the geometric interpretation of this is of course you know to think of this as the product of the magnitude of one vector times you know the projection of one on top of the other. So, this is you know easy enough to see once you are already you are already familiar with it I mean, but it is I guess it is not; it is not a priori obvious why this type of an operation should be the same in one direction or the other right you could as well do a projection in the other direction.

But there is a way to geometrically also see that you know if you were to project B on top of A or A on top of B, you would still get the same right, one way to see this is to just take the dot product between the directions. Then you have equal lengths, and then you can use a symmetry argument, and then the magnitudes simply will multiply right. So, that is one way to see why you will get the same answer if you project one vector on top of the other or if you were to project the other vector on top of the other first right.

So, there is another way of thinking of the dot product which also we are all familiar with which is to if you know the components right I mean I am thinking of three-dimensional vectors and this also is amenable to generalization to you know arbitrary dimension as long as they are Euclidean vectors right.

Then you can think of this as A dot B is A x B x plus A y B y plus A z B z plus more components, if there are more components right. So, let us keep this in mind as we generalize these ideas to you know more abstract vectors. They do not have to be Euclidean vectors, we have seen that you know the idea of vectors is much more general than you know these familiar kind of vectors right.

But before we generalize the notion of a dot product to you know the more abstract kind of vectors, let us look at how you know the fact that you have a you know this the notion of a dot product itself can be exploited in clever ways right. Again I am going to give you an example from high school geometry and there are many other similar problems which are elegantly solved using vectors.

So, let us look at one such example. So, the point is that when two vectors are orthogonal to each other, we exploit the fact that their dot product is going to be 0 right. So, the statement of the problem that we wish to prove is that you know if you drop a median of an isosceles triangle to the base, then that is going to be perpendicular to the base right.

Surely, you have seen the statement at some point of time and you must have used some pure geometry methods to prove this right. So, let us look at how with the aid of the dot product we can do it rather quickly and elegantly using vectors.

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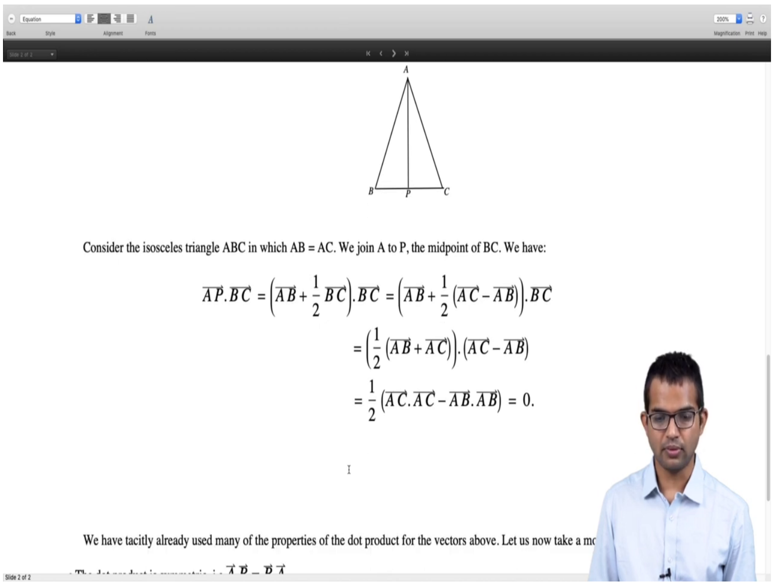

So, we have this triangle ABC right and I am joining. So, we are given the ABC is the isosceles triangle with AB equal to AC right BC is the base. And I am going to drop this median from A to the midpoint of BC right. And now let us look at the dot product between the vector AP and BC. So, I have AP dot BC is equal to AB plus half BC dot BC right. So, what is AP? It is AB plus half of BC right. But what is BC itself? I can rewrite BC as AC minus AB. So, I have AB plus half of AC minus AB, the whole thing is going to be dotted with the vector BC right.

So, then I can collect these terms together. So, I have AB plus half AC minus half B AB which is the same as half of AB plus AC. So, if you look at this you know carefully you see from symmetry what we are saying is this AP is nothing but half of AB plus AC which is how you would expect it to be right, so it is just because you have to drop this line from A to the midpoint of BC.

And then you also have dotted with AC minus AB. So far all of this would be true for any triangle; we have not exploited the fact that this is an isosceles triangle right, but we can go ahead and expand this. Then we have AB dot AC minus AC dot AB right so which will cancel, and then you have AB dot AC dot AC minus AB dot AB half of you know half of AC dot AC minus AB dot AB.

All of this up till this point is true for any triangle, but now comes the crucial point, which is that the magnitude of AB is the same as the magnitude of AC. So, these two terms will just cancel and then you get a 0 right, so that is a quick and elegant way to prove that. So, what we have managed to show is that this median AP is perpendicular to the base BC because the dot product of AP with BC is 0 right. So, that is what we managed to prove using vectors.

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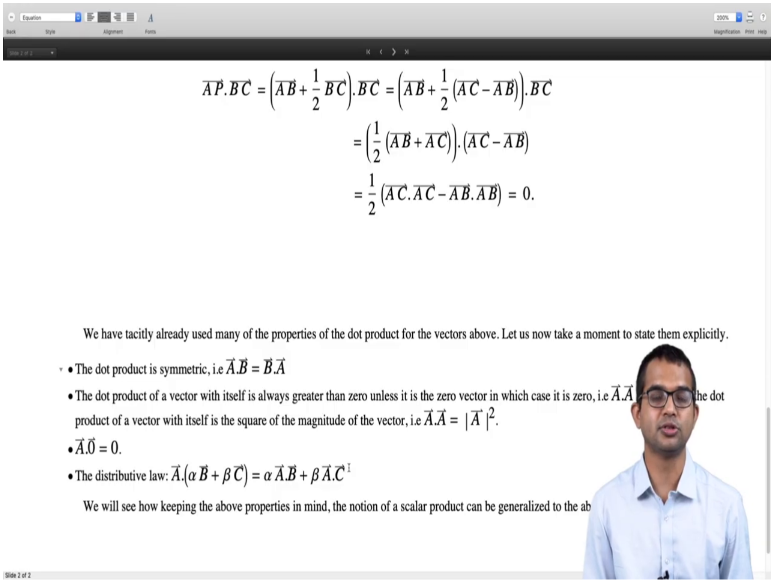

Now, in this discussion that we have had so far we have tacitly assumed many properties of the dot product. So, let us just summarize these properties right. So, in fact it is these properties that we are primarily concerned with because we are interested in how to generalize this notion of a dot product to a linear vector space right.

So, let us write it down right. So, first of all we have already repeatedly used the fact that A dot B is equal to B dot A right which is a symmetric property which we are so accustomed to so used to that we think that it is obvious right. But when we go to the linear vector space and how we can introduce the notion of a scalar product, there we will see that we have to tweak this a little bit, it is not so straightforward right. So, there is a modification that we will have to work with. So, let us keep this in mind.

So, the here for Euclidean vectors it is completely symmetric A dot B is equal to B dot A. And then the dot product of a vector with itself is always greater than zero unless it is the zero vectors itself if it is a zero vector then its magnitude is zero. So, the dot product of a zero vector with itself is of course, it is zero.

So, in general, the dot product of a vector with itself has a measure of the magnitude of the vector right. And it its better be a positive quantity. So, in fact, so we write this as A, so A dot A is always greater than or equal to 0; it is a number. And it is a real number which whose value is greater than or equal to 0 and we and so we use the square root of this A dot A as the magnitude. We know that it is the magnitude of the vector right. So, this is also one of the properties.

Then of course, if you take the dot product of any vector with the zero vector, you just get zero right so. And also an important rule that we tacitly assume in all these operations is the so-called distributive law right.

If you take some arbitrary linear combination of two vectors and if you take the dot product of some other vector with this linear combination of vectors, then it is going to be you know these coefficients will just come out, and then you can take these dot products with these various vectors that constitute this overall vector alpha B plus beta C. So, you get when you take a dot product with a you get alpha A dot B plus beta A dot C right.

So, all these properties we want to keep or the you know the best possible way in which you know these properties can be generalized we want to you know keep them when we go to when we define the notion of a scalar product with between vectors in a linear vector space which will be the subject matter of the next lecture.

Thank you.