**Mathematical Methods 1**
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**Linear Algebra**
**Lecture – 39**
**Diagonalizability of matrices**

Ok. So we have seen that, you know the matrices, you know can undergo similarity transformations and certain matrices you know there are special similarity transformations which it is possible to find for certain matrices, which make them diagonal, right. But we also said that not all matrices are diagonalizable. So, in this lecture, we will look at the conditions for Diagonalizability. When is a matrix diagonalizable? Ok.

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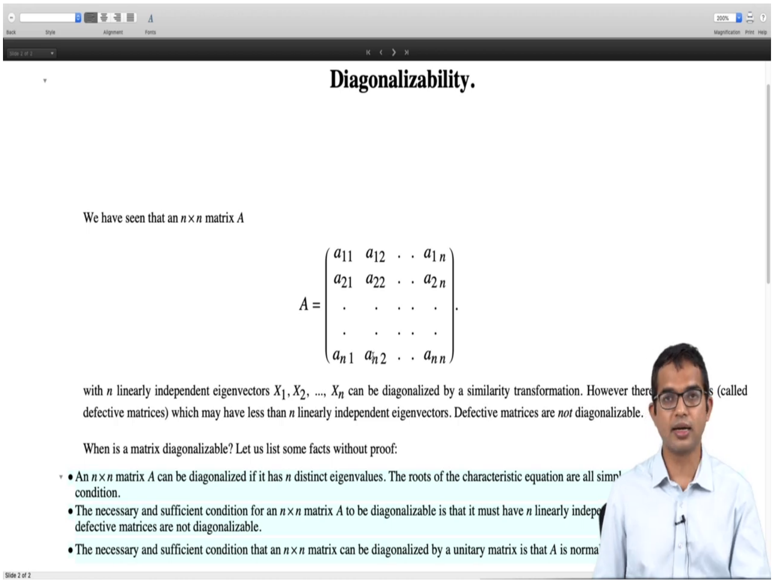


So, consider an n by n matrix A. So, you have all these elements of your matrix, right. If it has n linearly independent eigenvectors; it can be diagonalized by similarity transformation, right. But there are matrices which are not, which do not have n linearly independent eigenvectors right and they are called defective matrices. And so, defective matrices are not diagonalizable, right.

So, we would not go into a rigorous proof of any of these statements; but we will collect together a bunch of facts, which pertain to you know matrix diagonalizability, right. First of all there is a simple sufficiency condition; we have already seen that, if a matrix has all distinct eigenvalues, then for sure you know its eigenvectors are going to form a linearly independent set, guaranteed.

So, for sure any matrix which has you know distinct all eigenvalues distinct, for sure is going to be diagonalizable, right. So, this is a sufficiency condition. So, the roots of the characteristic equation in this case are going to be, you know, all roots are going to be simple roots.

So, there will be no repeated roots right; that is what this is, you know this is a sufficiency condition. Now, a general necessary and sufficient condition for an n by n matrix is that it must have n linearly independent eigenvectors. So, in other words it must not be defective, right.

So, the statement that a matrix is not defective is the same as the statement that it is diagonalizable. If a matrix is not defective, meaning it has n linearly independent eigenvectors; then we have seen you know by construction that you can put together, stack together all these linearly independent eigenvectors and diagonalize this matrix.

And it is also true that, if a matrix is diagonalizable; then it has [vocalized-noose] n linearly independent eigenvectors and if it has n linearly independent vectors, it is diagonalizable. Both these statements are the same right, you it is necessary and a sufficient condition.

Now, there is a class of matrices which is very important and appears a lot in quantum mechanics; for example, you know the class of matrices which are not only diagonalizable, but they are diagonalizable by a unitary transformation, right. So, the necessary and sufficient condition that an n by n matrix can be diagonalized by unitary transformation unitary matrix; you can find a unitary matrix is that A must be normal, right.

Again the proof of the statement is not necessarily very difficult; but it is somewhat technical and so we will skip this proof, right. So, we will just learn this as a statement that, if you have a normal matrix. So, what is a normal matrix? So, we already saw right; I mean the arguments which would go into the proof, some of which we have already seen.

Whenever you have a normal transformation, it means that if there are two distinct eigenvalues; then the eigenvectors corresponding to these distinct eigenvalues are going to be orthogonal to each other, right. So, that is the property, it is not, it is a stronger condition; it is not just that you have a bunch of eigen vectors which are linearly independent, but in fact they are going to be mutually orthonormal and you can always.

So, they are going to be orthogonal and then you can normalize them and so on, right. So, and you know the special case that one has to be careful with is when you have degeneracy, right. So, a normal matrix even if it is degenerate can never be defective right, that is what this is saying right; not only is it not defective, but in fact it is diagonalizable by a unitary transformation.

Meaning that, if you find all the eigenvectors, you can just put them together in a suitable orthonormal set and then you get a unitary transformation, right. So, it is not just a similarity transformation which diagonalizes it but it is a unitary transformation.

So, it seems like this condition for normality is a rather strong one; but these are very important, these are a very important class of operators and matrices, right. And a subclass of this is you know Hermitian matrices; we have already seen Hermitian matrices, where not only do you get a full complete set of eigenvectors, you know orthonormal and everything, but also the eigenvalues are real.

So, in fact unitary matrices are also another kind of normal matrices; we have seen this right, you can go back and review this. So, but the result I am just stating here is that, the necessary and sufficient condition which can be proven rigorously is that, an n by n matrix can be diagonalized by a unitary transformation, if and only if that matrix A is normal; meaning it commutes with its Hermitian conjugate.

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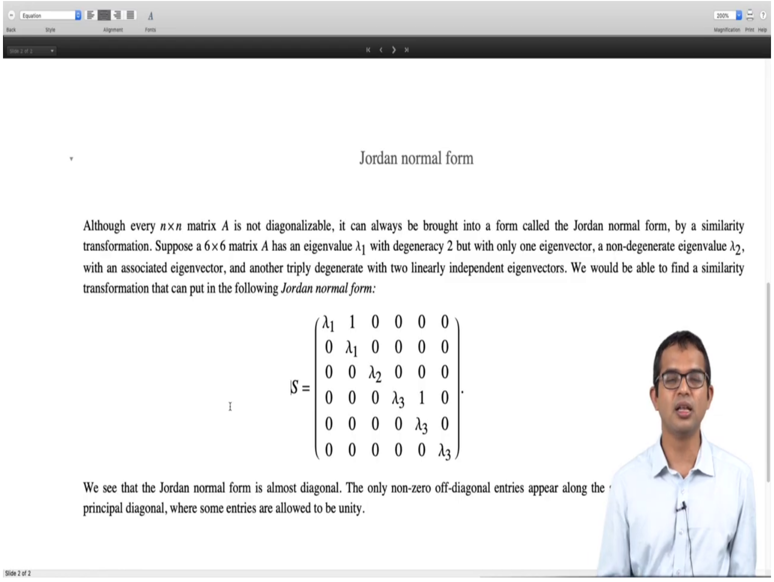


So, any matrix may you know can be brought into a form called a Jordan normal form; if you cannot diagonalize it, what is the next best you can do right; and that is what the Jordan normal form is, right. So, just in the interest of you know completing this discussion; we will include a very short description of what this is, we will not go into the details of how to actually implement this and so on. I will give it to you in the form of just one example.

So, any n by n matrix for sure can be brought into a form, like here where it is not going to be, it may not be diagonal, but it is almost diagonal. So, you have elements. So, the entire matrix is filled with 0s except along the diagonal for sure, that is where you have the freedom to put in the information; but also you allow yourself you know a bunch of 1s in this super diagonal.

So, this can be a 1, this can be a 1, you know all the elements in principle along this super diagonal closest to the just above the principal diagonal, you know you can allow for a few 1s, right. So, there is another way of thinking about this, you know think of this, consider a 6 by 6 matrix; I am looking at a 6 by 6 matrix, it has one eigenvalue lambda 1 with, but with degeneracy 2, it has degeneracy 2, but only one eigenvector right, you may have degeneracy 2 and 2 eigenvectors that is possible.

But if you have degeneracy 2 and only one eigenvector which you can find and then you have a non-degenerate eigenvalue lambda 2; lambda 2 is a safe eigenvalue. But there is a triply generated, triply degenerate eigenvalue lambda 3 right, but with only two linearly independent eigenvectors. Here you have the choice that you may have just 1 or 2 or 3; if it is all 3, then it is like you know it is like a usual case.

But I will just illustrate what happens, you know in this somewhat complicated case; I have taken this to be triply degenerate, but let us say you have two linearly independent eigenvectors, then we will be able to put it in this form. There is a similarity transformation you can find, such that you can put it in this form.

The first block corresponds to the eigenvalue lambda 1 and lambda 1; since you can find only one eigenvector, you know 1 will appear here, if you could find 2, then you could also make this also 0. The second eigenvalue is there is just a single eigenvalue; so of course it is just lambda 2.

Now, the third eigenvalue, you know the first pair has only one eigenvector right, you have you can form a block like this; if you had you know all three eigenvalues had only one eigenvector, then you would think of this as a 3 by 3 and then you put another one here, you can bring it in that form.

But on the other hand you have, you know an eigenvector corresponding to these two eigenvalues is a single one. And there is another eigenvector corresponding to lambda 3, which you can think of as for the purposes of this type of a you know treatment, you can think of this as another eigenvalue; since it has its own eigenvector right, although you have lambda 3, lambda 3 and lambda 3, right.

So, this is the next best you can do and the theorem states that you can always do this; no matter what your matrix is like, it can be defective. Of course, if it is not a defective matrix, we all know what the Jordan normal form is; the Jordan normal form is what we call the diagonal matrix, right. We have proved a stronger result, you do not need these 1s; they are all 0s and you can bring it to the diagonal form.

And the Jordan normal form is the next best you can do, which is almost diagonal and it has you know it has its usefulness in certain contexts. And primarily its importance is when you are trying to understand how general these results are, right. But for our purposes we have to be aware that, you know there are certain matrices which are defective and then you may not be able to diagonalize them.

But, you know the most you know the matrices that we will commonly work with particularly in physics are matrices which are diagonalizable. And when they are diagonalizable, they are you know this procedure has been laid down and we will see briefly what advantages diagonalization gives in the next discussion.

Thank you.