Mathematical Methods 1
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Linear Algebra
Lecture – 38
Diagonalization of matrices

Ok. So we have seen similarity transformations. So, in this lecture, we will look at a special kind of similarity transformation which takes a matrix and which you know makes it diagonal right, so that it goes by the name of Diagonalization of matrices ok.

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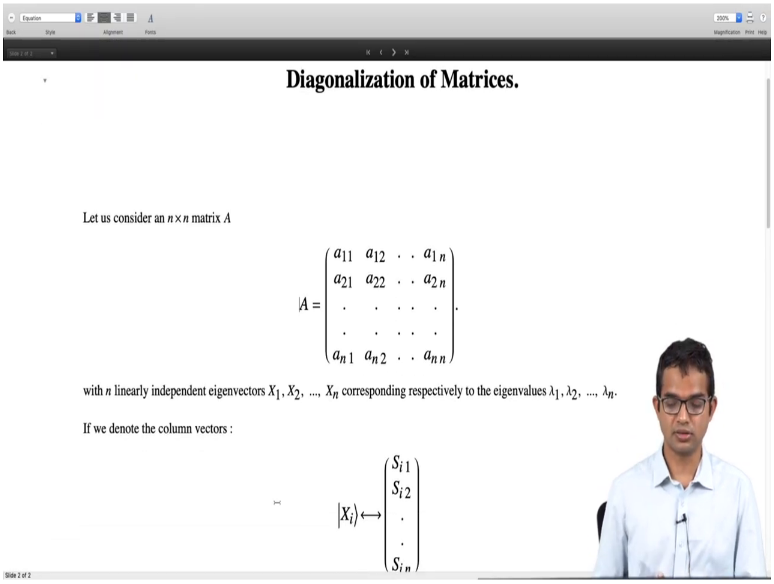

Let us consider an n by n matrix A. You know I am explicitly writing it out, it has all these you know elements of this matrix, there are n squared of these. So, I am looking at a matrix which has n linearly independent eigenvectors right. I am not saying it is linearly independent column vectors that is a different concept; I am talking about a matrix with n linearly independent eigenvectors right.

So, if you recall from several lectures ago, we looked at certain matrices which we called defective matrices right. Although every matrix is guaranteed to have exactly the number of eigenvalues as its dimension. Although some of these eigenvalues may be repeated, there is no guarantee that you know every matrix will have you know eigenvectors which will complete the space right. You there are matrices which are called defective matrices, but we will you know talk about this a little bit later.

But for the purpose of this lecture, let us look at matrices which have n eigenvalues lambda 1 all the way up to lambda n right. So, some of these could be repeated, there is no problem with that. But the key requirement is that you know all these n eigenvectors are linearly independent. If this happens, then let us collect all these eigenvectors.

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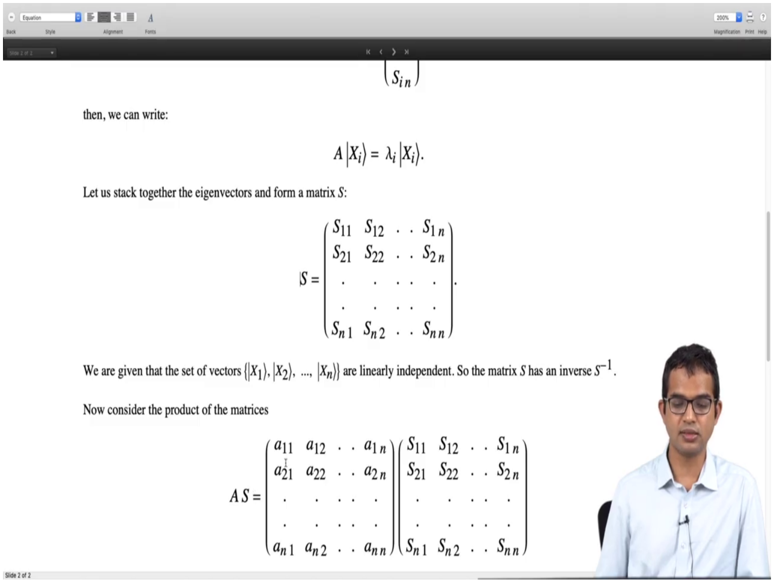

So, you have n linearly independent eigenvectors. We will notate this as you know X i these coefficients S i 1 S i 2 so on all the way up to S i n. Then we have this defining property, these being eigenvectors. These are A acting on X i should give you lambda i acting on X i.

So, if you stack them all together right, so now you see the point of the previous lecture where we looked at properties of matrices formed by staking together such linearly independent vectors. We saw that these are all linearly independent column vectors.

So, you have n linearly independent column vectors, each of these column vectors is linearly independent. Therefore, necessarily S is an invertible matrix. So, this is important for us as you can see. So, let us consider its inverse and denote it as S inverse as usual. Now, the product of these matrices A times S is important.

So, A times S is you know there is this matrix A this is what we started with, and then we just stacked together all its n linearly independent eigenvectors to form this matrix. So, we can look at the product of these two matrices.

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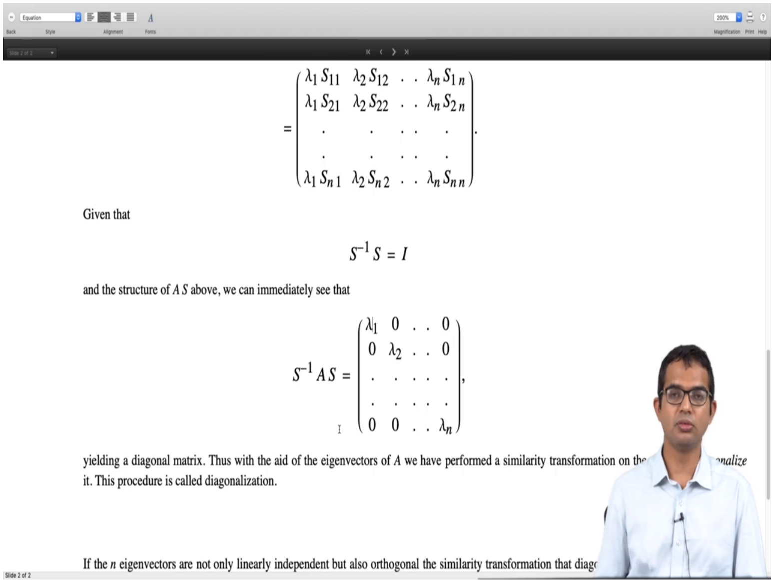

And then we see that in fact, this product is something that we can work out in a straightforward way because what is the first what is the effect of multiplying this matrix with the first column vector; if I look at this column vector it is really a acting on X 1, so that should giving you lambda 1 X 1, and so I can write lambda 1 X 1 as you know lambda 1 S 1 1 lambda 1 S 2 1 all the way up to lambda 1 S n 1.

And similarly if I take this entire matrix and multiply by the second column, I must get back the same column, but now with these factors lambda 2 in here, lambda 2, lambda 2 so on.

And likewise for every column and for the final column, you see that if I multiply you know this entire matrix with this column the nth column S 1 n, S 2 n all the way up to S n n will give me back the same column except that I have to tag along these factors lambda n. So, A S is something that I can immediately workout because of what I have chosen S to be.

Now we also know that S inverse exists. So, S inverse times S is equal to I. So, let us sandwich A between S inverse and S. So, let us workout S inverse A S. So, S inverse S is equal to I. So, S inverse A S, so we see that when we do A S you basically get s, but every column has an extra lambda 1 there right you can convince yourself that all these does is if I multiply S inverse with A, A S instead of S inverse with S, I will still have a diagonal structure.

But now I will have to put in lambda 1 here, lambda 2 here, all the way up to lambda n right. This you can see because S inverse acting on S is I. So, if you take the first row and multiply by the first column, it must give you just 1, but now I have a lambda 1 here. So, it will give you lambda 1, so likewise lambda 2 and so on up to lambda n.

So, what you have done just now is something called diagonalization right. So, we have managed to do a similarity transformation on our original matrix A. And we have obtained a diagonal form for which all of whose elements are just the eigenvalues of your matrix A right. So, if we could find such a similarity transformation, you could use this to get to the eigenvalues of this matrix.

But oftentimes you know to find the eigenvalues of the matrix, you can just go to the characteristic you know equation you know come up with the polynomial find its roots, you have another way of finding the eigenvalues right. But diagonalization involves finding this similarity transformation which puts your matrix in a very nice form right.

We will see later that this has important applications, diagonalization of a matrix when it is possible to diagonalize a matrix you know gives you some is a very nice thing to do. Because you can compute lots of properties of your matrix which are in a matrix are understood in terms of you know the eigenvalues, but also these the similarity transformation which diagonalizes the matrix also contains the lot of information because all the eigenvector information is also enshrined in it.

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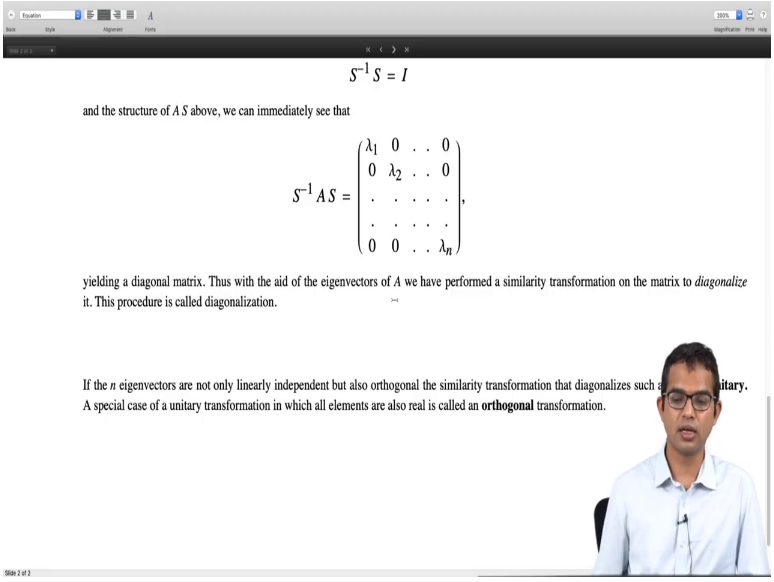

So, there is a, so we have talked a lot about unitary matrices right. So, if your n vectors are not only linearly independent, but also orthogonal. Then in place of S inverse A, S you will just have you know U dagger A U right. So, if in addition to, if your eigenvectors of your original matrix you had not only X, A X i giving you lambda i A X i with all these X i being linearly independent right.

There is a you know shorter class of matrices whose eigenvectors are not only linearly independent, but they are also orthogonal to each other. We have noted some of these matrices right. And they appear in many important contexts like in quantum mechanics right, Hermitian operator for example right. And or there is a way to find if in the presence of degeneracy it is still possible to find the certain vectors which are eigenvectors of your matrix and which are orthonormal.

When you have such a scenario not only do you have a similarity transformation, but you have a special kind of a similarity transformation that is a unitary transformation which imposes some more constraints on the nature of you know a nature of your the final diagonal matrix of your system right. So, unitary transformation is a special class of similarity transformation.

And there is another kind of transformation which is a special kind of unitary transformation, and that is called an orthogonal transformation when you are dealing with these real numbers right. So, U dagger is the same as just U transpose if all the elements of your matrix are real, then you get an orthogonal transformation which is another further special class of unitary transformations or unitary matrices ok. So, that is all for this lecture.

Thank you.