Mathematical Methods 1
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Linear Algebra
Lecture – 35
Eigenvalues and eigenvectors: useful results

Ok. So, we have seen how the theory of linear vector spaces with finite dimensions is basically the same as the theory of matrices right. And so, in this spirit we have been looking at a number of results of matrices, and we will continue along this direction; in this lecture we will collect together a bunch of useful results, pertaining the Eigenvalues and eigenvectors of matrices ok.

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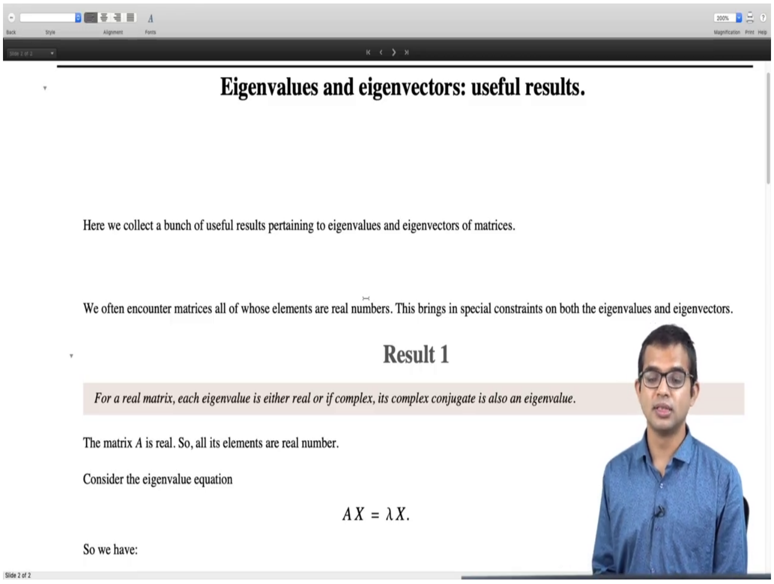

So, it is very common to find matrices which have only real numbers right. So, if you have only real numbers in a matrix you know, if you have matrices with only real numbers it brings in some constraints for the eigenvalues and eigenvectors. So, let us work this out.

So, if you have a real matrix, each eigenvalue is necessarily either real or if it is complex, then its complex conjugate is also an eigenvalue right. So, complex eigenvalues appear in conjugate pairs right. So, this is a consequence of you know the ok, let us work out the argument for this. So, the matrix A is real, so all its elements are real. We start with the eigenvalue equation A acting on X is lambda times X.

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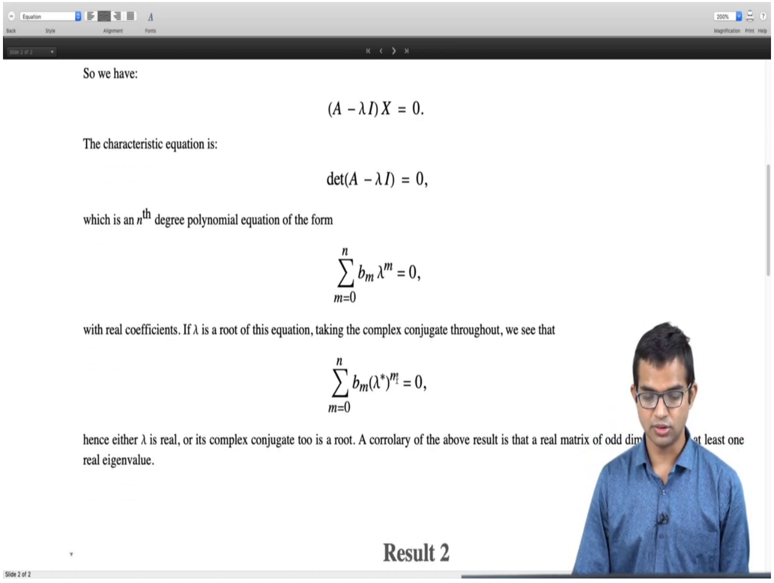

So, the characteristic equation is just the determinant of A minus lambda I equal to 0, which we know is an n th degree polynomial equation of the form summation over i b m times lambda to the m. So, this should actually go all the way from 0 right. So, m equal to so, if m equal to 0 that is the constant right you can have a constant term, then you have you know a linear term a quadratic term so on all the way up to n right.

So, the point is that all these coefficients are real, that is the key point fine, and whenever you have a polynomial equation of this kind. Let us say that lambda is the root of this equation, then it means that you know this equation holds and then we have the freedom to take the complex conjugate of this entire equation. And then, since b ms are all real so, they remain unchanged and so, wherever you have lambda you have to replace it with lambda star.

So, which means that you know the same equation holds, so let me correct this. So, this is m going all the way from 0 to n in place of lambda I have lambda star. And hence either lambda is real or its complex conjugate to it. So, it means that either lambda equal to lambda star, or lambda star which is distinct from lambda is also a root right.

So, the corollary of the above result is that if you have a real matrix of odd dimension, then at least it has to have one real eigenvalue right. Because, if you have an odd number you cannot have only pairs of complex, complex numbers and their conjugates that will be at least one which does not have a pair and that eigenvalue must be real ok.

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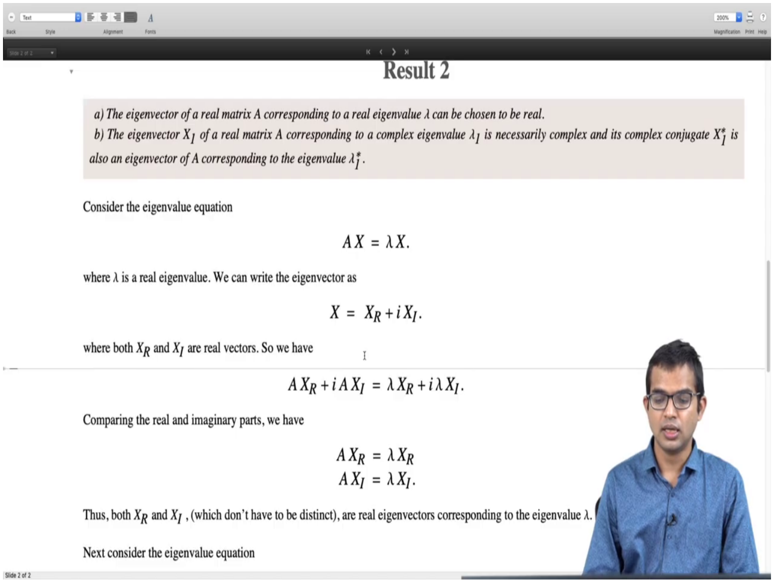

So, the next result is that if you have a real matrix, then you know and if it has a real eigenvalue, then the eigenvector corresponding to this real eigenvalue can be chosen to be real right. That is the first result I mean of this. Second results the part A of this result. And, then the other result which we show is if you have a real matrix; that means, all the elements are real.

And, if you have an eigenvalue lambda 1, which is complex and then it goes with an eigenvector X 1. Then, you can show that the eigenvector corresponding to lambda 1 star, which is also an eigenvalue right which we have already argued the complex conjugate of X 1 is going to be the eigenvector corresponding to lambda 1 star right. Again this is something that we can show from the first principles.

So, we have A X equal to lambda X so, let us say that you know we take. So, there is a real eigenvalue lambda, corresponding to which there is an eigenvector X. Right at this point we are not saying that X is real or complex. But, let us say it is complex in general you have managed to find an eigenvector.

Then, you can of course, right this complex vector as you know real part of this plus i times imaginary part right, you have two vectors X R and X I the real part of X will you know all the real parts of these components will go into X R and the complex the imaginary part will go into X I both X R and X I are real vectors right. So, now, you will show that in fact, both X R and X I themselves are eigenvectors with the same eigenvalue right.

So, the argument is very straightforward; so, you just see that A times in place of X you put this in so, you see that A X R is plus i times A X I is equal to lambda X R plus i times lambda X I, now you just compare coefficients compare the real part and the imaginary part on both sides.

So, since A is real X R is real A times X R better be real and A is real X I is real so, A times X I is real. So, that means in fact A X I is the imaginary part on the left hand side. And again lambda X R is real lambda X I is real. So, i times lambda X I is you know lambda X I is the imaginary part of the right hand side.

So, comparing the real and imaginary parts, we immediately have A X R is equal to lambda X R and A X I is equal to lambda X I. Therefore, both X R and X I are real eigenvectors corresponding to the eigenvalue lambda; of course, X R and X I do not have to be distinct right.

They could be basically the same or you know the product of if there is a constant factor or something like that, there can be two of them, which can be you know trivially connected, or there may be independent also right depending on the situation right. So, the point is that, if you have or are able to find one eigenvector for real eigenvalue that is the way to, just take the real part or take the imaginary part of this vector. And, you have a you know real eigenvector corresponding to real eigenvalue.

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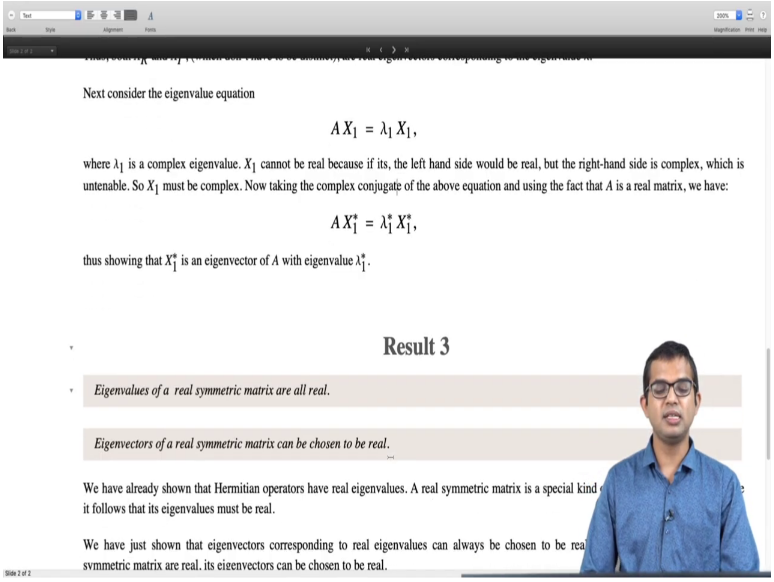

And so, the other result is you know if A X 1 is equal to lambda 1 X 1 and lambda 1 is a complex eigenvalue. So, if A is real and if X 1 also were real, then the left hand side would be real whereas the right hand side is complex which is untenable. Therefore, you know X 1 has to be complex and then you just take a complex conjugate of this entire equation on both sides.

So, the complex conjugation of A will do nothing to it. So, you will be just left as A, but X 1 will become X 1 star lambda 1 will become lambda 1 star X 1 star. But, this is explicitly just the eigenvalue equation you know saying that the matrix A has i X 1 star as an eigenvector with eigenvalue lambda 1 star, which is the result we are trying to prove ok.

The next result is that eigenvalues of a real symmetric matrix are all real and eigenvectors of real symmetrics, symmetric matrix can be chosen to be real. And these are two results you know two parts of the same result.

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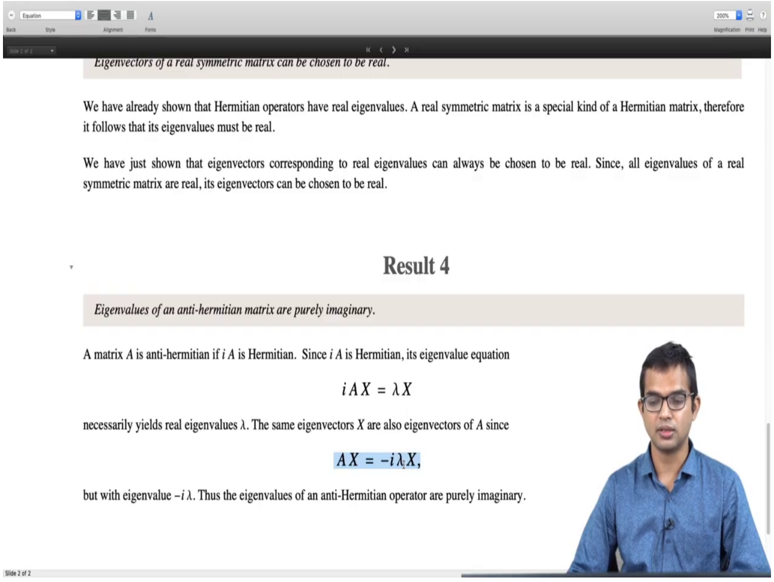

But, I mean we have already seen that Hermitian operators in general right, have real eigenvalues. We proved this in greater generality right. Where the way we did it was you know to look at what Hermitian operator does to some arbitrary Cat vector and then we took the dual space vector of this.

And, then use the Hermicity property and then manage to show, you know, compute the same matrix element in two different ways and then argue that. This forces all eigenvalues of a Hermitian matrix to be real. And therefore, a real symmetric matrix is a special type of a Hermitian operator right, where all the elements are also real.

Symmetric property of you know this real matrix is where the Hermiticity comes in. And so therefore, it also follows that its eigenvalues must be real right, you can also you know rigorously go for the same type of procedure when you take a dual space vector and so on and directly prove it. But, we have already proved this in greater generality.

So, indeed real symmetric matrices have real eigenvalues. You know and the other result is also something that we have just proved right, that we have just said that whenever you have a real matrix with real eigenvalues you can always find a way to get real eigenvectors. And so, since for a real symmetric matrix all eigenvalues are real, there is a way to find real eigenvectors corresponding to them right.

So, this is also something which follows from our earlier discussion. And, the final result we want to discuss in this lecture is that the eigenvalues of an anti Hermitian matrix are purely imaginary. An anti Hermitian matrix is a matrix such that i times this matrix is Hermitian right. So, another way of defining this is A dagger is equal to minus A right.

So, since i A is Hermitian if I look at i A right, we have just shown that the eigenvalue equation for i A will necessarily give you real eigenvalues. So, if i A X is equal to lambda X, where lambda is necessarily real I can pull this i to the right hand side and then I am I have minus i on the right hand side so, I have A X is equal to minus i lambda X.

So, this again is an eigenvalue equation for the matrix A right, which is Hermitian. So, this is so, this suggests that and with eigenvalue minus i lambda where lambda is real right which means that every eigenvalue of a Hermitian anti Hermitian operator. So, A is anti Hermitian, whereas i A is Hermitian right.

So, we started with the assumption that A is anti Hermitian. So, i A is Hermitian so we have managed to show that the eigenvalue equation for the anti Hermitian operator A you know leads to purely imaginary eigenvalues right. So, this is something which is a direct consequence of the definition of anti Hermitian matrix and the properties of Hermitian matrices.

So, that is all for this lecture we will look at more results pertaining to matrices in the next (Refer Time: 12:02).

Thank you.