Mathematical Methods 1

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

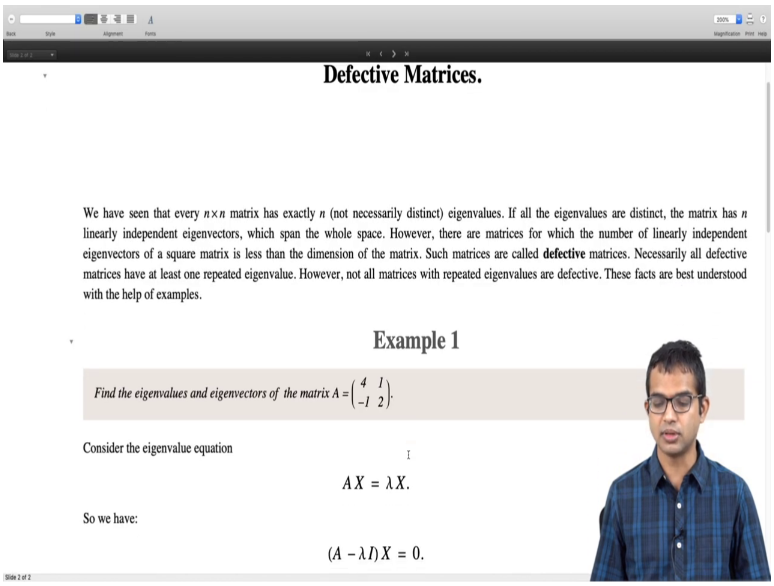
Linear Algebra
Lecture – 34

Defective Matrices

So, we have seen that given a square matrix, you can write down an eigenvalue equation and you know going to the fundamental theorem of algebra, we will always be able to find you know as many eigenvalues as the dimension of the matrix. Sometimes some of these eigenvalues may be repeated, but the eigenvalue equation always yields n roots.

But on the other hand, you know there are cases where matrices may not have may not may not give you know an enough eigenvectors to span the overall space right. So, the best way to understand this is with the aid of some examples which we will look at in this lecture ok.

(Refer Slide Time: 01:13)



So, you know matrices whose eigenvectors are unable to span the overall space, they are called defective matrices right. If you know the number of linearly independent eigenvectors of the square matrix is less than the dimension of the matrix right, such a scenario appears when you have a defective matrix right.

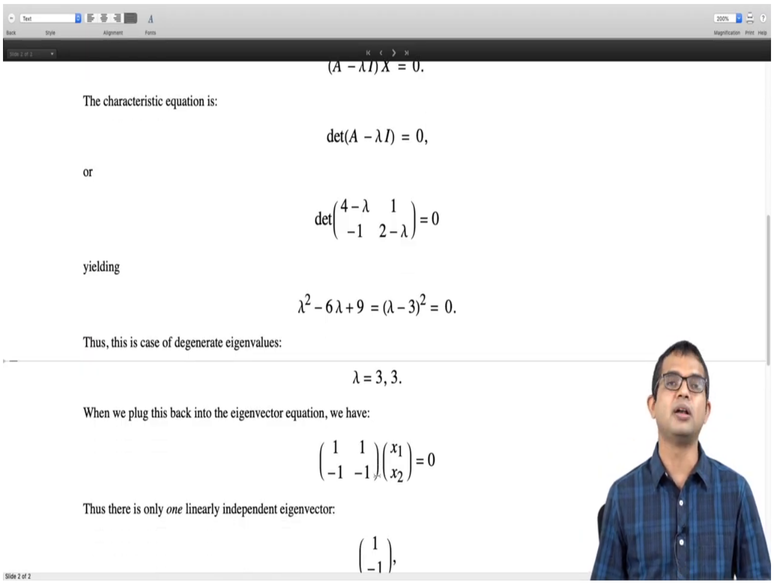
So, necessarily all defective matrices have at least one degenerate eigenvalue – repeated eigenvalue right, because we have seen that you know corollary to one of the theorems which we discussed in the previous lecture is that if you have all non-repeated eigenvalues.

If you have distinct eigenvalues, then you are going to have so many distinct and linearly independent eigenvectors. And if you have n eigenvectors which are linearly independent that are going to span the whole space. So, there is no question of such a matrix being defective right.

So, defective matrices happen only when there is degeneracy, but not all degenerate matrices will be defective right. So, we will just look at some examples of matrices. You know two examples we are going to look at one where degeneracy is a you know result of the defectiveness of the matrix, and the other cases you know where you can have a degenerate matrix but which is not defective ok.

Let us look at the first example which is you know this matrix A equal to 4 1 minus 1 2. So, the eigenvalue equation is AX is equal to lambda X. So, we have A minus lambda I acting on X is equal to 0.

(Refer Slide Time: 02:52)



So, the characteristic equation is the determinant of A minus lambda I equal to 0, which is the same as saying determinant of 4 minus lambda 1 minus 1 to minus lambda equal to 0. So, you expand this out. You know I have lambda minus 4 times lambda minus 2 minus minus 1, so plus 1, so that will give you lambda squared minus 6 lambda plus 9.

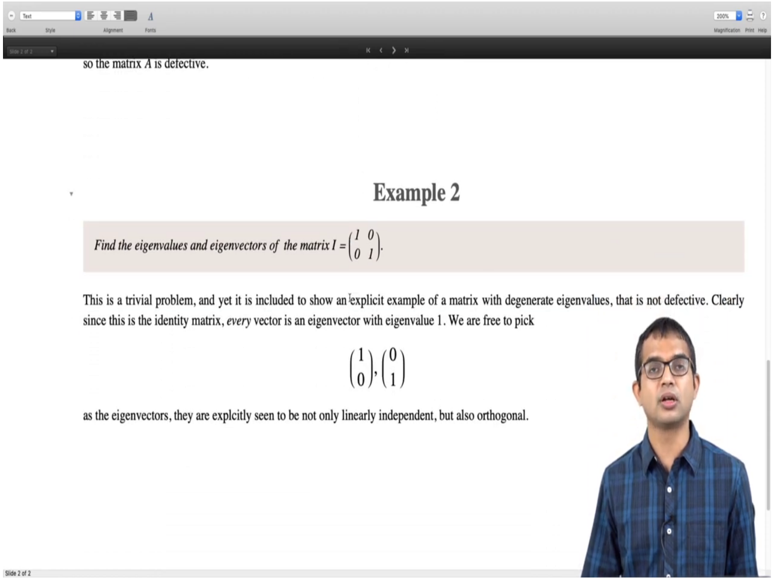
But we see that this is the same as lambda minus 1 lambda minus 3 the whole square equal to 0. So, there you see that the roots of this equation are in fact 3 and 3. So, you have a repeated root or you know repeated eigenvalue, degenerate eigenvalue.

So, now let us see what happens to the eigenvectors. Let us try and find the eigenvectors you know corresponding to the degenerate eigenvalues. Are there two independent eigenvectors and can we find them, let us look at them. So, if we plug this back into this equation right, so this equation.

So, we write x as x 1, x 2. So, we have 1 1 minus 1 minus 1 x 1 x 2 equal to 0. So, you see that the first equation is x 1 plus x 2 equal to 0, and the second equation is minus x 1 minus x 2 equal to 0, minus x 1 minus x 2 equal to 0.

So, basically both equations are the same right. So, there is no separate content coming from the second equation. Therefore, in fact, you will be able to find only one independent linearly independent eigenvector right. 1 minus 1 that is the only eigenvector that you can find from this from the characteristic equation. So, this means that the matrix is defective right. It is a 2 by 2 matrix, but there is only one eigenvector and there is a repeated eigenvalue which is 3 right.

(Refer Slide Time: 04:22)



So, now let us look at another example which is actually a kind of a trivial example of a matrix. Again we will stick to a simple 2 by 2 matrix which is degenerate and which is not defective right. If you want to pause the video and try to come up with your own example, please feel free to do so.

So, the hint is that it is extremely simple. In fact, you might say it is trivial after you look at the answer. So, find the eigenvalues and eigenvectors of the matrix I equal to 1 0 0 1 right. So, this is completely trivial because this is the identity matrix, and there is nothing to do. You do not have to even come up with the characteristic equation or anything right.

So, every vector, every normal vector is an eigenvector of the identity matrix right, but it is a two-dimensional space, so but there are you will only be able to find two linearly independent eigenvectors, where if you take a third one then that is going to be you will have a linearly independent set.

So, I mean we are free to pick any two linearly independent vectors and that is going to, you know, suffice because they are going to be eigenvectors. But let us say we just pick 1 0 and 0 1, they explicitly seem to be both linearly independent and orthogonal. All linearly independent vectors do not have to be orthogonal, but orthogonal vectors are linearly independent right.

So, in this case, I am just picking 1 0 and 0 1, randomly you may try out some other and that also is going to work out. So, the point is that there is a trivial example of a matrix which admits degeneracy, but which is not defective right. So, in general you can have n by n matrices you know which maybe of either of the above kinds, you can have you know higher order degeneracy.

You may have three eigenvalues which are repeated, and then you may have to closely check whether it is a degeneracy of this kind which means that there is no defect in the matrix, and then you will still be able to find eigenvectors which will span the space.

But on the other hand this is a more sinister kind of degeneracy, when this happens a matrix becomes defective and so the eigenvectors of such matrices are unable to span the space right. So, in quantum mechanics, we do encounter operators which have degeneracy. But there is always this kind because we work with Hermitian matrices, where we will see that you know Hermitian matrices always are able to give you eigenvectors which will span the overall space right.

So, the point of this lecture is to illustrate that although you know the eigenvalue equation is straightforward and any matrix will necessarily have an eigen values, although some of them may be repeated, the number of linearly independent eigenvectors that matrices have can you know depends on you know the particular type of matrix that you are looking at.

And in particular, there are these very pathological matrices which are called defective matrices where the number of linearly independent eigenvectors may be less than the dimension of the matrix, and one has to be careful about such matrices. So, that is all for this lecture.

Thank you.