Mathematical Methods 1
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Linear Algebra
Lecture – 33
Eigenvalues and Eigenvectors of matrices

Ok. So, we have seen that finite dimensional linear vector spaces, you know, have representations, all operators which live on finite dimensional linear vector spaces have matrix representations. So, if you understand matrices well then we understand finite dimensional vector spaces.

And so, therefore, we are going to look at a number of properties of matrices starting from this lecture which is about Eigenvalues and Eigenvectors right; certain properties of eigenvalues and eigenvectors will be discussed here of matrices. And then, you know more properties in the context of eigenvalues and eigenvectors will come up later, but let us look at a few very important results in this lecture, ok.

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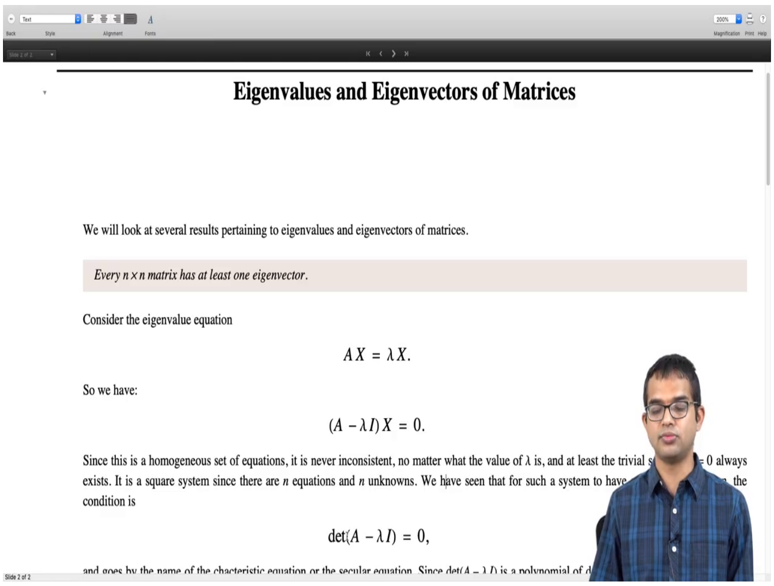

So, we start with this crucial result that every n by n matrix has at least one eigenvector, right. So, how do we find eigenvalues and eigenvectors? We start with the eigenvalue equation, right. So, you have A times X is equal to lambda X. So, we need to find you know vectors X and we need to also find the scalar lambda, which is in general a complex number, right. So, we need to find a way to compute lambdas and the X corresponding to those lambdas.

So, what we are really interested in is finding a solution to this equation, right, A minus lambda I acting on X equal to 0, right. This is a; this is a matrix which operates on a vector on a column vector and it gives you 0, right. So, this is if you recall from many lectures ago described using linear equations, right. So, this is really a set of homogeneous equations and therefore, it can never be inconsistent because all the stuff on the right hand side is 0.

Now, the only question is does it have only the trivial solution or does it have non-trivial solutions, right. So, we have seen that the condition for the non-trivial solution to exist, you know when you have n equations n unknowns, right. So, we are looking at a square matrix. So, the number of rows in this equation is the same as the number of unknowns, right.

So, we have exactly the situation Taylor made where we had a determinant condition for this, right. For this to have a non-trivial solution the trivial solution is of course, the case where you take all you know this vector X to be 0 and in which case you do not call it an eigenvector, it is a null vector, right. So, for it to be an eigenvector you must have a non-null vector that is what we are looking at, that is what we are interested in, right.

So, the point is that this n by n matrix has at least one such genuine eigenvector. So, it is a square system and it has that means, it has n equations and n unknowns. So, this means that for this to have a non-trivial solution we must demand the determinant of A minus lambda I equal to 0, right. And this is what is called the characteristic equation corresponding to this matrix.

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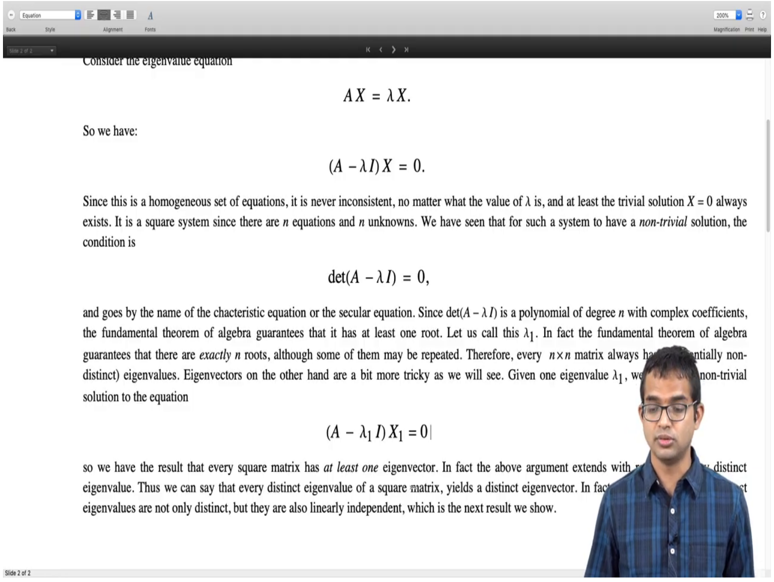

And now this is a polynomial determinant of A minus lambda I is a polynomial of degree n, right. So, this comes from the property of determinants; you can try to expand the determinant of an n by n matrix and then you will see that if there is a coefficient like this. For sure it is going to be a polynomial of degree n with complex coefficients, right in general.

And so, there is this fundamental theorem of algebra which guarantees that a polynomial of degree n with complex coefficients has at least one route, right. And let us call this lambda 1. So in fact, actually the fundamental theorem of algebra implies that there are going to be exactly n roots for an nth order polynomial nth degree polynomial. And although some of these roots may be repeated, right.

So, you will be able to factor this polynomial as lambda minus lambda 1 into lambda minus lambda 2 you know, so you can in general write it as lambda minus lambda 1 to the power k 1 times lambda lambda minus lambda 2 to the power k 2 and so on, all these parameters k 1, k 2 and so on must add up to n. So, that is a consequence of the fundamental theorem of algebra.

So, there is at least one group. So, let us just call that lambda 1, right. So, there would be only one route if all the routes are repeated. You know if you have a scenario like lambda minus lambda 1 the whole power n then there is only one route. But in general you know you could have more routes, ok. So, it has at least one eigenvalue and that is lambda 1, right. In fact, it has n eigenvalues, some of which could be repeated, right. So, let us call this root that we have as lambda 1.

Now, given one eigenvalue lambda 1 we can go back and plug this in here, right. So, we have to find a solution to this homogeneous equation which is definitely possible. So, you can find an X 1 such that A minus lambda 1 I times X 1 is equal to 0 and you will find a non-trivial solution, right. Therefore, you will find a non-trivial vector corresponding to a certain eigenvalue right, lambda 1; and therefore, you have at least one eigenvector, every matrix has at least one eigenvector.

So, in fact, you can make a stronger statement here. In fact, every non-repeated eigenvalue will yield a distinct eigenvector, right. So, the same argument will hold. So, in place of lambda 1 you have, suppose you have a lambda 2 which I managed to find, these are all roots of this polynomial equation.

So, if there is another root then you can go back and plug this in into your homogeneous system of equations and you know our understanding from systems of linear equations guarantees us because the secular equation or the characteristic equation holds guaranteed that you will be able to find a non-trivial solution for X.

So, you will get a distinct eigenvector corresponding to a different lambda 2 and different lambda 3 and so on, right. So, there are as many distinct eigenvectors as there are distinct eigenvalues for any matrix.

So, in fact, we can make a stronger statement which is the next result, which is that the set of distinct eigenvectors which are derived from distinct eigenvalues are all necessarily linearly independent. Not only are they distinct eigenvectors, but they are also linearly independent, ok.

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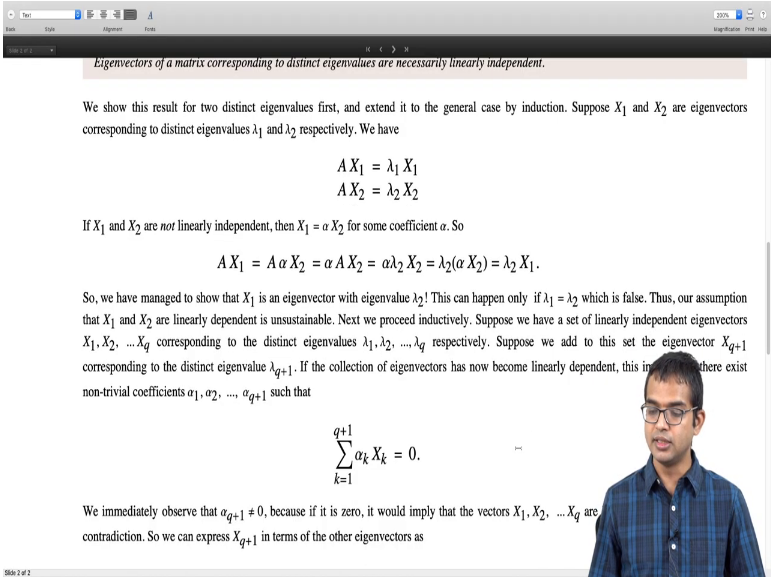

So, eigenvectors of a matrix corresponding to distinct eigenvalues are necessarily linearly independent. To show this let us start with just two distinct eigenvalues lambda 1 and lambda 2 and then we will proceed by induction, right. So, we have lambda 1 and lambda 2 and 2 eigenvectors X 1 and X 2. So, we have A acting on X 1 is equal to lambda 1 X 1, A acting on X 2 is equal to lambda 2 X 2.

So, we want to show that if lambda 1 is not equal to lambda 2, then X 1 and X 2 are linearly independent, right which means that suppose X 1 and X 2 are not linearly independent, right, so that would imply that X 1 is equal to alpha times X 2 for some coefficient alpha.

If this holds, let us see that there is a contradiction. If A X 1, then we let us operate with A on X 1, so you have A X 1 is equal to A times alpha X 2 because X 1 is equal alpha X 2 that will give us alpha times A X 2 which is the same as you know A X 2 is lambda 2 X 2.

So, you have alpha lambda 2 X 2. Now, if I exchange these coefficients you know the order and then I have lambda 2 times alpha X 2, but alpha X 2 is the same as X 1. So, I have managed to show that A X 1 is equal to lambda 2 X 1, right. But we know that A X 1 is lambda 1 X 1, right. So, what we managed to show is that X 1 is an eigenvector of A, but with eigenvalue lambda 2.

Now, but we have you know we have the you know this part of the proposition that lambda 1 and lambda 2 are distinct eigenvalues, right. So, which is impossible and there is a contradiction with what we have just shown which is that the eigenvector X 1 has eigenvalue lambda 2, right. So, there is no way that this statement and you know A X 1 is equal to lambda 2 times X 1 can be consistent with lambda 1 not equal to lambda 2.

So, the only way you know this can be avoided is if our initial assumption is wrong. So, X 1 and X 2 are not linearly are not linearly dependent. So, X 1 and X 2 are linearly independent, right. So, we have managed to show it for just two distinct eigenvalues and then by induction we can show that this holds also for you know an arbitrary number of distinct eigenvalues, right.

So, let us suppose that you have distinct eigenvalues lambda 1, lambda 2, all the way up to lambda q and let us say X 1, X 2, all the way up to X q are linearly independent eigenvectors, right. So, now, we will show that if we put in one more eigenvector X q plus 1 and corresponding to a distinct eigenvalue lambda q plus 1. Necessarily this new set X 1 all the way up to X q plus 1 also must be linearly independent, right.

To see this, suppose we make the assumption that the moment you add X q plus 1, the set is not linearly independent. So, that means, the set is linearly dependent. So, we will be able to find some non-trivial coefficients alpha 1, alpha 2, all the way up to alpha q plus 1 such that you know this relation holds summation over alpha k times X k is equal to 0.

Now, immediately we can observe that the q plus 1 coefficient, alpha q plus 1 cannot be 0, right. If it is 0 then it would imply that these vectors X 1, X 2 all the way up to X q are linearly dependent. So, this would be a contradiction. So, we must have that this coefficient is nonzero, and so we will be able to express you know X q plus 1 in terms of the other eigenvectors.

So, we will be able to divide throughout by this coefficient and then we have X q plus 1 is equal to minus 1 summation over these you know constants times X k, where now k runs from 1 to q with at least one of these coefficients which is nonzero, right.

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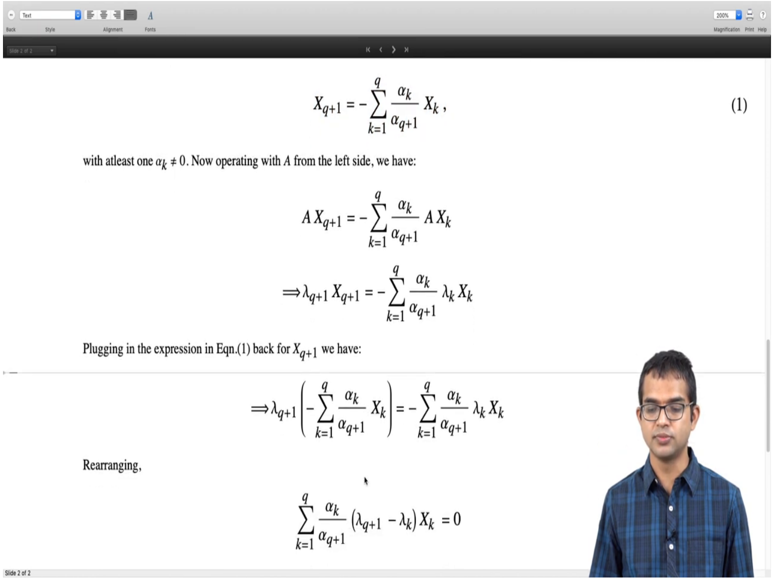

So, now if we operate with A from the left side; so, we have A acting on X q plus 1 is equal to you know minus summation over you know all these coefficients times A acting on X k, but A acting on X q plus 1 is the same as lambda q plus 1, q plus 1, right. So, and then again here in place of A X k you can replace it with lambda k, X k.

And then in place of X q plus 1, we will again plug back this equation, right. We have already written X q plus 1 in the X k's. So, in place of X q plus 1 we will plug in this expression and then we have you know this expression follows from plugging in place of X q plus 1 you know this equation. And then, if we rearrange all these terms we are able to write this as a summation over k of a bunch of coefficients, you have alpha k alpha q plus 1 times you know this difference of these eigenvalues times X k is equal to 0.

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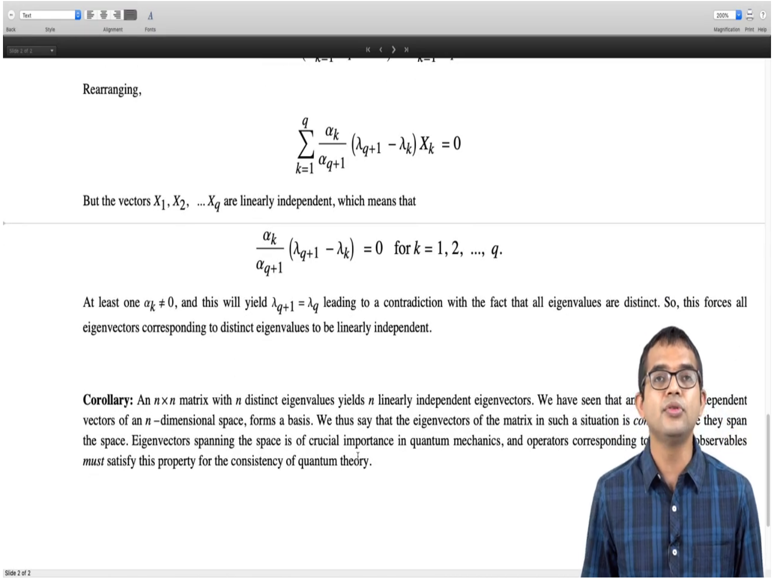

Now, what do we have here? We have the sum of some coefficients times this vector X k is equal to 0, but X k you know k going from 1 to q are linearly independent. So, that means that the only way that this can happen is if all of these coefficients are 0, right. And you know from here we know that at least one of these coefficients alpha k is nonzero because if all of these were 0, then there would be a contradiction to this earlier proposition.

So, therefore, at least one alpha k is nonzero and that would immediately imply that you will be able to find some you know lambda q plus 1 must be equal to some one of those lambda k's, for k going from 1 to q, right. So, it is not a very difficult argument, you just need to go over it carefully and then you can convince yourself that if we make the assumption that you know X 1, X 2 all the way up to X q are linearly independent.

You know that implies that X 1, X 2 all the way up to X q plus 1 also must be linearly independent because this is going to lead you to a contradiction. It is not sustainable with the condition that all these eigenvalues have to be distinct. So, we have managed to show that there is a contradiction with that assertion, right.

So, the only way that this could have been avoided is if we had not made this initial assumption which is that you know this box of vectors X 1, X 2 all the way up to X q plus 1 is linearly dependent. So, they have to be linearly independent; and so that is all, that is the result.

So, what we have managed to show is if you have a matrix with a bunch of distinct eigenvalues then each of these distinct eigenvalues gives you a distinct eigenvector, but not only a distinct eigenvector, but the set of eigenvectors obtained from these distinct eigenvalues is in fact, going to be a linearly independent set.

And this has consequences when all the n eigenvalues of your n by n matrix are distinct, right. So, what we can immediately tell is that if you have an n by n matrix with n distinct eigenvalues, then you have n linearly independent eigenvectors. And so, we have seen this result many lectures ago that if you have n linearly independent vectors in an n dimensional space then they form a basis, right.

So, here we have; so, we can take all these n linearly independent eigenvectors and form a basis which means that any vector in the space can be expanded in terms of these n linearly independent eigenvectors. So, this is a scenario which is called you know it is a completeness condition, right. So, matrices whose eigenvectors span the whole space are complete, right.

So, this is of crucial importance in quantum mechanics. Any physically meaningful observable you know there is an operator corresponding to an observable and every such operator corresponding to a physical observable must yield eigenvectors which span the whole space. If it does not then you know there is going to be an inconsistency called quantum mechanics. And that is why we work with Hermitian operators, right.

So, real quantities correspond to operators which are Hermitian as you might have seen in quantum mechanics. And so, we have already seen that Hermitian operators have this property of completeness that comes about because of linear independence of various eigenvectors.

So, we will return to this in the context of matrices a little while later. But the main take home message from this lecture is that if you have an n by n matrix for sure it has n eigenvalues, although some of these eigenvalues may be repeated. But to be sure that you know you have n linearly independent eigenvectors, right, so that is not always the case we will look at more examples of this later on.

But in this lecture, we have shown that if you know many distinct eigenvalues each of these distinct eigenvalues will give you a distinct eigenvector which also forms a linearly independent set. And the particular case of interest here is if you have an n by n matrix with n distinct eigenvalue. Then, for sure your matrix will yield n linearly independent eigenvectors which can form a basis for your space. That is all for this lecture.

Thank you.