Mathematical Methods 1
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Linear Algebra
Lecture – 30
Normal operators

So, we have seen how Hermitian operators and unitary operators both these classes of operators you know, yield the second result: corresponding to each of these, the eigenvectors of distinct eigenvalues have to be orthogonal right. So, this is a crucial property.

So, you can put together a bunch of orthogonal eigenvectors and create a basis and that is going to span the whole space right. So, that is you know it's of great importance so, this you know gives you matrices which are diagonalizable and so on which we will come back to at a later time.

So, we might be interested in the question: what is the most general class of operators? So, why should we force operators to be Hermitian or unitary or you know some conditions have to be imposed on operators - not all operators satisfy this type of property.

And so, if we ask the question, what is this, what is the general condition that operators must satisfy such that you know this property, the second property which is that any two eigenvectors corresponding to distinct eigenvalues are going to be orthogonal.

It turns out that this kind of a class is called you know normal operators which is a much bigger class of operators in which you know Hermitian operators and unitary operators are also part of this bigger class right. So, that is the topic of discussion for this lecture ok.

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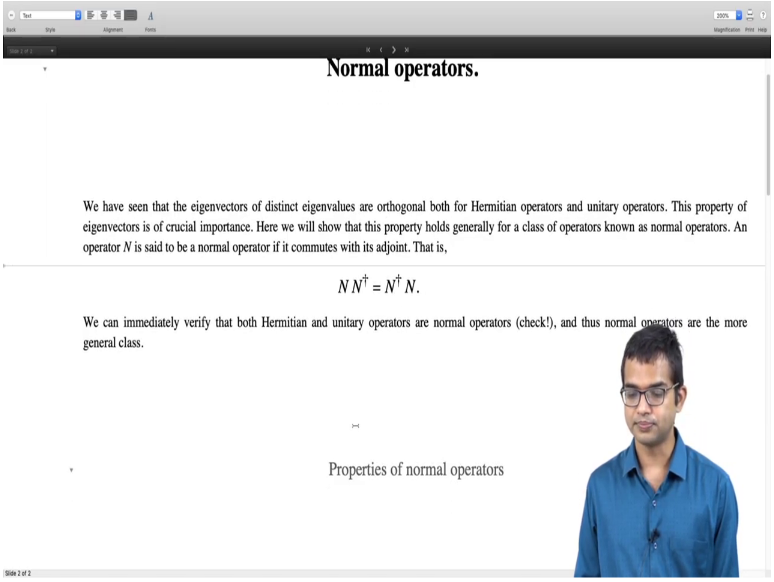

So, an operator is called normal if it commutes with its adjoint right N N dagger is equal to N dagger N right. So, this operation N times N dagger minus N dagger N is called the commutator and if that commutator is 0 right which is the same as saying N N dagger is equal to N dagger N, then we say that such an operator is normal right. You can immediately check that both unitary operators and Hermitian operators are normal operators right.

So, Hermitian operators clearly N dagger it is going to be N itself and N dagger will be N so, then N squared is of course, equal to N squared and the defining property of a unitary operator is u u dagger is equal to u dagger u is equal to you know identity right. So, it turns out that you do not need this to be equal to identity. If you just manage to show that a N N dagger is equal to N dagger N that is already enough right.

So, these are a broader class of operators, you know it is already enough as far as you know this property that we are going to prove is concerned which is about the second property related to eigenvectors right. If it also is equal to identity, we have seen that it is the class of unitary operators and that gives you unique modularity for the eigenvalues.

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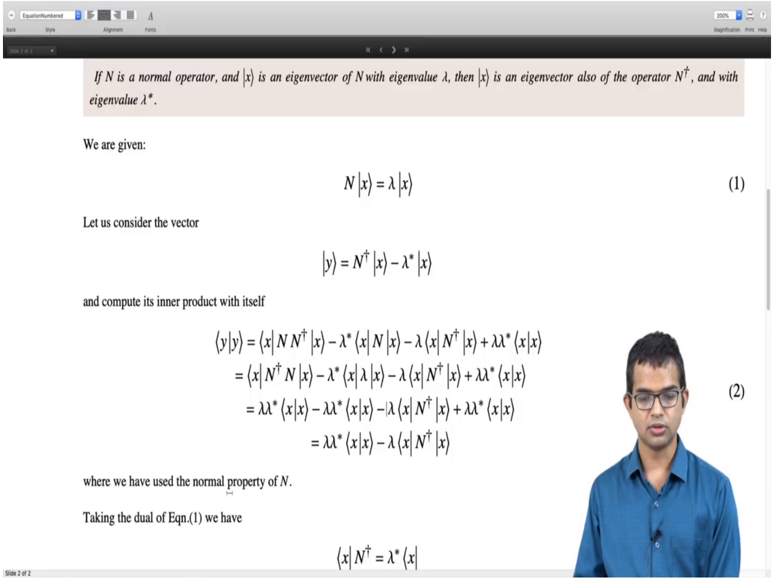

Let us look at this result for eigenvectors of normal operators, but before we go there, there is one more result which we can prove for normal operators. If N is a normal operator and x is an eigenvector of N with eigenvalue lambda, then x is an eigenvector also of the operator N dagger and with eigenvalue lambda star right.

So, we have given N acting on x is equal to lambda x. So, we want to argue that in fact, x is an eigenvector also of another operator which is the adjoint of N and whose eigenvalue is also something we can work out and that is going to be lambda star right. How do we see this? So, the argument is actually quite clever right. So, we already have a priori knowledge of this lambda star being an eigenvalue of N dagger and with whose eigenvector is also known.

So, let us just simply consider this vector y, N dagger you know is an operator it and x is a vector so, N dagger acting on x is a vector and we have the freedom to form another vector which is you know subtracting minus lambda star acting on this vector x. So, now we will show that this is in fact, the 0 vector right. So, how do we do this? We just simply compute the inner product of this vector y with itself. If you do this carefully and you see that you know you get four terms right.

So, the first term is x acting on N N dagger x right so, this is something that you should convince yourself right. So, if you take the bra vector corresponding to N dagger x, that is going to give you x N. Then, you have N dagger x, then you have a minus lambda star x so, I have taken the you know the bra vector corresponding to this x and then x, then you have a minus lambda x N dagger x plus lambda lambda star x x right. So, there are four terms.

And then, you see that two of them immediately simplify. So, you have x N dagger x so, so, I am using the normal vector property. So, N N dagger is the same as N dagger N so, I have the freedom to change the order, then I have minus lambda star. So, N acting on x is just lambda times x. So, it is the eigenvalue equation, then I have minus lambda x N dagger x, I just leave this as it is plus lambda lambda star x x.

Then, but again N acting on x is lambda x so, this N dagger x will give me lambda star. So, I have lambda lambda star inner product of x with x minus again this lambda will come out lambda lambda star inner product of x with x, then I have, I leave these last two terms as it is. So, you see that the first two terms get cancelled so, this will go away to 0, these two terms so, then you are just left with lambda lambda star x x minus lambda times this matrix element x N dagger x.

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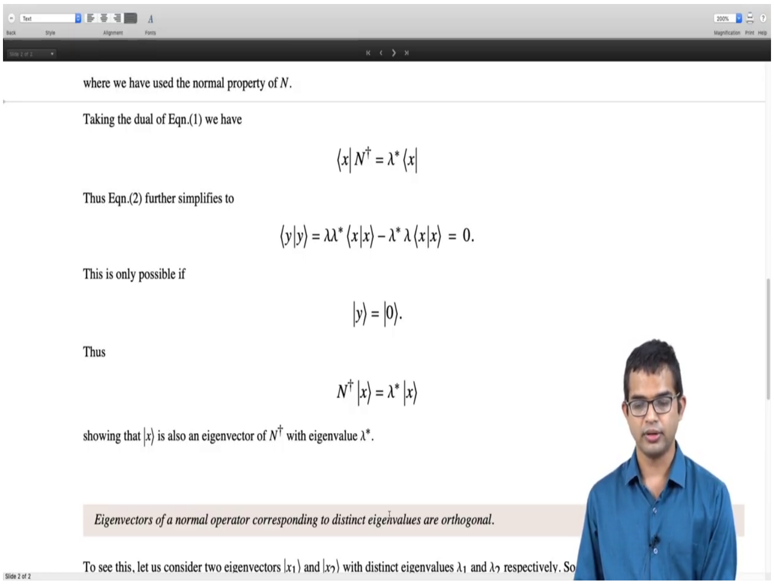

So, now we will use this property of you know we will take the dual of this normal vector N x. So, we have x N dagger is equal to lambda star x right. So, then, there is equation 2. So, then we further simplify to the inner product of y y is lambda lambda star x x minus so, then I am bringing the ket vector here from the right side. So, x x N dagger x is equal to lambda star x x so, I have minus lambda lambda star x x, but both these terms are the same and there is a negative sign here. So, the inner product of y with y is 0.

So, what we have managed to show is that this is basically the 0 vector, y is there is 0 vector because there is only one vector whose inner product with itself is 0 right and that is the 0 vector. And that immediately means that N dagger acting on x is equal to lambda star x which is what we set out to prove right.

So, x is if x is an eigenvector of N with eigenvalue lambda, it implies that x is also an eigenvector of N dagger, but with eigenvalue lambda star right. We will use this result to prove the main result which we are after, which is that Eigenvectors of a normal operator corresponding to distinct eigenvalues are orthogonal.

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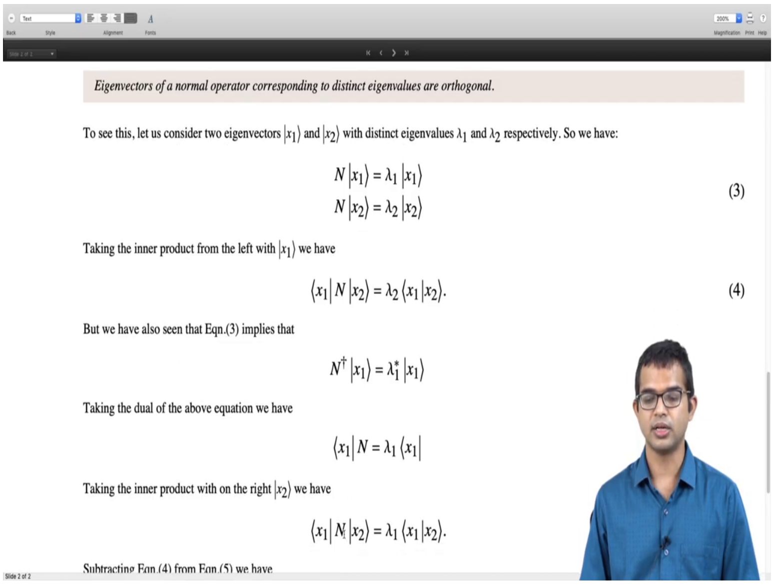

So, let us consider two eigenvectors x 1 and x 2 with distinct eigenvalues lambda 1 and lambda 2 right. So, you have N acting on x 1 is lambda 1 x 1 and N acting on x 2 is lambda 2 x 2. So, if you take the inner product from the left with x 1 right, we have x for the second equation right, we have x 1 and x 2 is equal to lambda 2 x 1 x 2.

But we also seen that you know from this equation 3 implies N dagger x 1 is equal to lambda 1 star x 1 right because x 1 is an eigenvector of N with eigenvalue lambda 1, x 1 is also an eigenvector of N dagger, but with eigenvalue lambda 1 star.

So, now if I take the dual of this equation I have x 1 bra x 1 N is equal to lambda 1 bra vector x 1 and then, I can bring in the ket vector x 2 from the right hand side x 2 from right hand side, I have x 1 N x 2 is equal to lambda 1 x 1 x 2.

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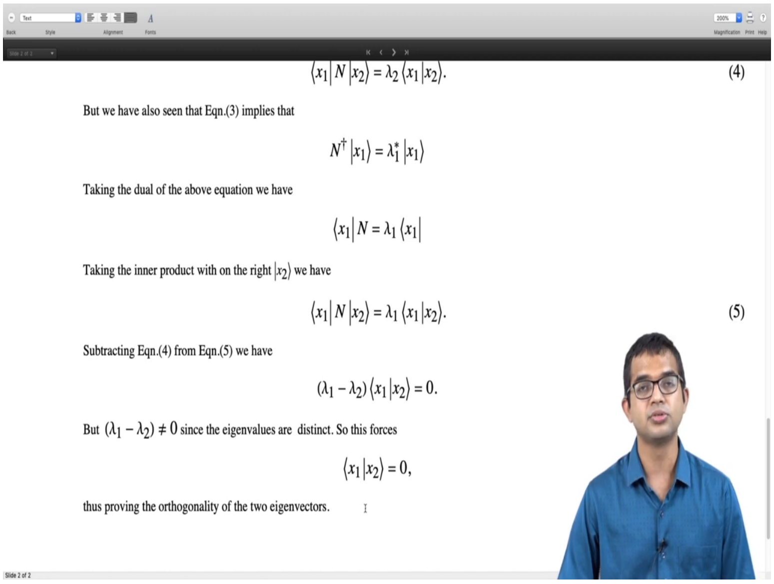

So, if I look at equations 4 and 5, I have managed to compute this matrix element x 1 N x 2 in two different ways and I have found that one of them is lambda 2 times x 1 x 2, the other one is lambda 1 times x 1 x 2.

If I subtract these two, I get lambda 1 minus lambda 2 times x 1 x 2 is equal to 0, the inner product of x 1 x 2 is equal to 0 and the only way this is possible since we have taken lambda 1 and lambda 2 to be distinct is if the inner product of x 1 and x 2 itself is 0 that is proving the orthogonality of two eigenvectors with distinct eigenvalues right.

So, this is a general result, we could have started from this and then, argued that both Hermitian operators and unitary operators are normal operators and therefore, this results holds in each of these cases, but it is instructive to see to individually apply the properties of each of these kinds of operators and then, you know generalise and a consequence of this property is that all normal operators are you know diagonalizable when we go to matrices.

So, we will discuss just like we have Hermitian operators, there are Hermitian matrices, there are the unitary matrices, there are normal matrices and so on right. If you have an N by N normal matrix so, we will argue later on that such matrices are diagonalizable which means that they have a you know as many eigenvectors as you know as many linearly independent eigenvectors as the size of the matrix and in fact, they can span the whole space right.

So, this is an idea which is best illustrated with examples right, we will do in the course of you know lectures that will follow. At this point, I just want to point out that all the results we have shown so far apply to general linear operators right at for a general linear vector space right. So, later on we will specialise to matrices and you know look at some of these consequences right. That is all for this lecture.

Thank you.