Mathematical Methods 1
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Linear Algebra
Lecture - 03
Linear Vector Spaces: Immediate Consequences

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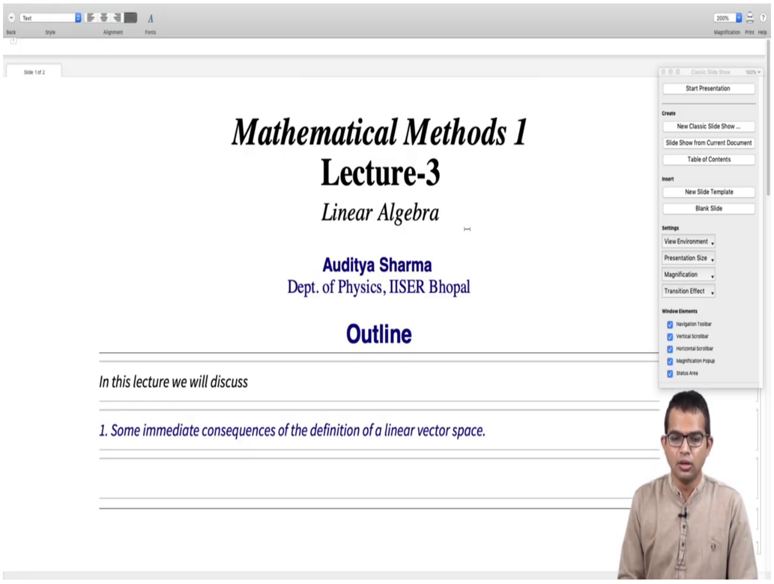

So, in this lecture we are going to look at some Immediate Consequences of the definition of a Linear Vector Space right. So, we defined what a linear vector space was in the previous lecture, and some of these properties will immediately give us some constraints - that is what we are going to look at in this lecture ok.

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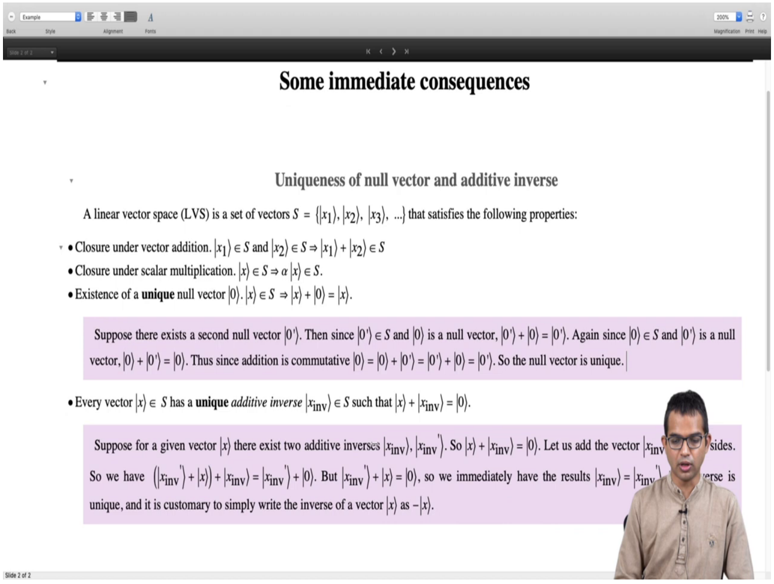

So, we said that a linear vector space is a set of these abstract vectors and it must satisfy certain properties; one is closure under vector addition, then there is closure under scalar multiplication. So, then we also said that it has a null vector right.

So, now, we want to argue that it automatically implies that this is going to be a unique null vector right. So, the definition does not by itself you know explicitly it does not have to forbid the presence of more than one null vector. But, we will see that it is impossible for you to have more than one null vector, there is going to be one null vector you can actually prove it right.

So, many a time you will see in textbooks where this is taken to be a requirement right. So, that there is a unique null vector itself is part of the definition, but we just want to point out that it is not essential to encode it into the axioms, it is something that can be derived right.

So, how does this come about? Right. So, suppose there exists a second null vector 0 prime right. So, that is the argument. So, let us concentrate on why there is a unique null vector. Suppose there is a second null vector 0 prime. Then, since 0 prime is an element of S, the property of a null vector is that if you add x to 0 you must get back x.

So, now 0 prime is an element of S and 0 is a null vector. So, 0 prime plus 0 must be equal to 0 prime right. So, 0 prime is this vector from the space and it has remained unchanged when you have added the null vector to it. But on the other hand 0 itself is an element of this vector space and 0 prime is a null vector.

So, if you add the null vector to this to any element of this space 0 plus 0 prime you must get back the same vector that you started with which is 0. So, since addition is commutative right. So, 0 is equal to 0 plus 0 prime which we have just shown here and whether its 0 plus 0 prime or 0 prime plus 0 prime 0 prime plus 0, which is given to be this 0 prime, it's the same and therefore, 0 equal to 0 prime right.

So, it just follows from commutativity that you cannot have more than one null vector, there is a unique null vector in your vector space. And likewise you can argue that the additive inverse of any vector is also unique right. So, we have said that every vector of your vector space has an additive inverse and that now we are going to say that we do not have to explicitly specify that its unique, but it has to be unique it follows from you know the; you know the structure of the all these properties already.

Suppose let us so, what is the proof for this? Suppose there exists two inverses let us say; x inverse and x inverse prime I am going to call them. So, for a given vector x, there are two additive inverses x inverse and x inverse prime. So, x plus x inverse must be equal to 0 by definition of the additive inverse.

So, let us add this x inverse prime to both sides. So, I am going to add from the left side. It does not matter because addition is commutative, but so let us say I add from the left side and then I will use the associativity of addition. So, I have x inverse prime plus x plus x inverse is equal to x inverse prime plus 0 right. So, because on the right hand side also I have added x inverse prime.

But what is x inverse prime plus x? I can combine the first two objects. So, but this is going to be x inverse prime plus x also must be 0 right this is by definition I have said x inverse prime is an additive inverse of x. So, x inverse prime plus x must be equal to 0.

So, we immediately have the result x inverse is equal to x inverse prime. So, the inverse is unique and it is customary to simply write the inverse as minus of this vector right. So, sometimes the minus sign is put inside the ket, but it's simplest to just write it as minus the vector x and then you explicitly see that the addition of x plus minus x is equal to 0.

So, the vector 0 right I mean to if you want to be very precise it must be written as you know x plus additive inverse is equal to the vector 0 right, it is a vector 0, but oftentimes it is just written as equal to 0 right just the scalar 0. And so, the reason for this is the following: so, we will show one more consequence of this.

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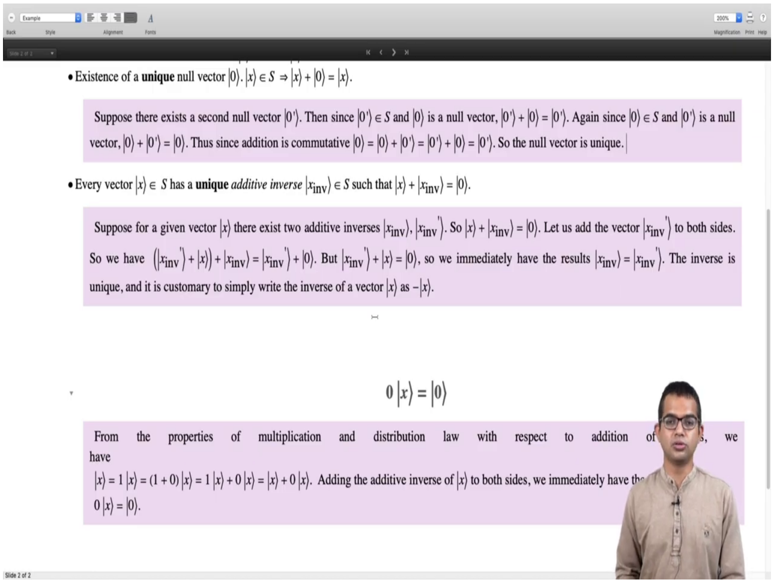

So, you can show that if you multiply any vector 0 with any vector x with the coefficient 0 you are going to get the vector 0. So, in some sense if you so, the analogy is with you know 2 d vectors if you take a vector and multiply it with the scalar 0, you are going to you know annihilate any notion of direction with it right.

It is the same whether you multiply a vector which is pointing along the x axis by 0 or whether you multiply a vector which is pointing along the direction 45 degrees to the x axis with 0 you are going to just get a 0 right.

So, but it should I mean technically speaking it is a 0 with the x component is 0 and the y component is 0, but it is you know we would usually just write this down as 0. The null vector 0 is often just written as the scalar 0 and you have to understand from the context that is what is being referred to ok.

So, let us prove this so we can prove. So, we had this condition where we wrote down explicitly one property of scalar multiplication, which is that if you multiply any vector with the scalar 1 you are going to get back the same vector. So, sometimes you know this property is also sort of encoded into the properties of addition multiplication of vectors. But, this also is something that we can prove directly from the distributive law.

So, why not you know keep the basic set of requirements or axioms as minimal as possible right. So, let us give out this proof. So, this is so, the idea here is to show you how some of these abstract proofs work out right. So, the properties of multiplication and the distribution law with respect to addition of scalars, we have you know if you take any vector x is equal to 1 times x. So, this is part of the rule that is already given to us it's part of the properties of scalar multiplication.

But what is 1? 1 is the same as 1 plus 0 right, this is the addition of scalars 1 plus 0. And then we have this distributive law 1 plus 0 times x is the same as 1 times x plus 0 times x. And then which is the same as again, we use this rule that 1 times the vector x is equal to x which is equal to and then you have plus 0 times x.

So, now what we have managed to show is that any vector x is the same as the vector x plus 0 times vector x. Now, we can you know add this additive inverse on both sides, vector x added to its additive inverse on both sides from the left side if you wish it does not matter the order because, this operation is commutative you will get a cancellation right x minus x will give you the vector 0 the null vector 0. And we immediately have the result 0 times x is equal to vector 0 right.

So, you might wonder why we are working very hard to prove things which seem totally obvious right. So, the objective here is to show you how you know how you know it is like an exercise in rigor, how you can start with some very small set of axioms.

And then work out you know proofs of statements which look obvious, but the point is to show you that you know even these kinds of assertions can be proved based on some very

very elementary set of axioms right. I mean the whole course we are not going to take this approach right. So, we have a physicist proof approach to a number of problems that will encounter as we go along.

But, these are you know simple to understand and it they have their own beauty and that is why we have we are including some of these arguments right at the beginning so, we show with the help of some very very basic set of definitions which go into you know what a vector is and what are vector spaces.

We can show that there are these inbuilt constraints, which make both the null vector and the additive inverse of these vectors unique. And also this property of 0 times x is equal to the vector the null vector 0 also comes out.

Thank you.