Mathematical Methods 1
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Linear Algebra
Lecture – 29
Unitary operators

So, we have seen how Hermitian operators satisfy two crucial properties. And in this lecture, we will look at another important class of operators called unitary operators. These two have a couple of very important properties which we will prove in this lecture, ok.

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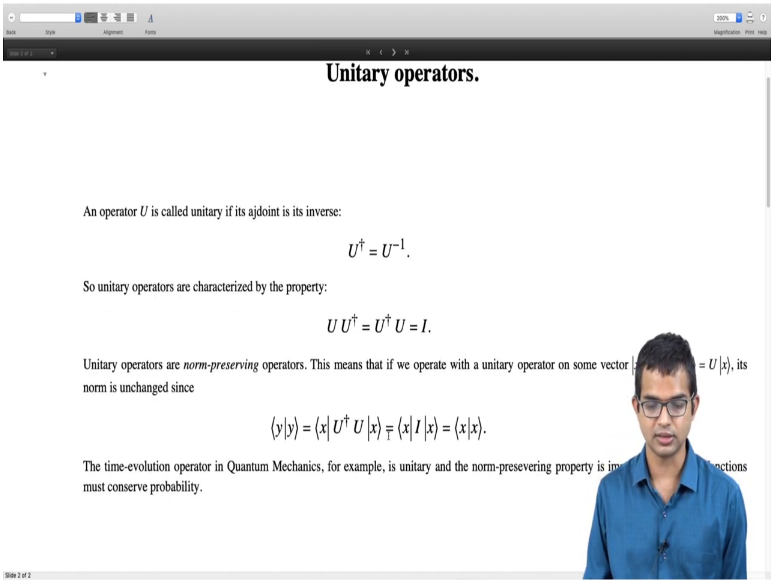

An operator is called unitary if it is adjoint is its inverse, right. A Hermitian operator is Hermitian if its adjoint is itself, but a unitary operator is unitary if its adjoint is its inverse. So, U dagger is equal to U inverse, right.

So, unitary operators are characterized by the property, U times U dagger is equal to U dagger times U is equal to I. And unitary operators are what are called norm preserving operators. So, this means that if you operate with a unitary operator on some vector x, its norm is going to be unchanged.

So, the inner product of x with x is the same as the inner product of y with y as we can immediately show. So, let y, where y is you know U acting on x, right. So, the inner product of y with y is you know the inner product of U x with the dual vector which is going to be x U dagger as we saw, right in the previous lecture. So, you have x U dagger U x, but U dagger U is the same as I and I does nothing, so you have x x. So, the inner product of y y is the same as the inner product of x x. So, the norm is unchanged.

And so, one classic example of a unitary operator is the time evolution operator seen in quantum mechanics, where the unitarity is crucial for and its norm preserving properties is crucial because probability needs to be conserved, right. So, this is one context in which unitary operators appear.

So, let us look at how it is like the Hermitian operators you know satisfied two properties one with regard to eigenvalues and one other regard to Eigenvectors corresponding to distinct Eigenvalues. We will look at you know somewhat similar properties, but with you know with a slightly different flavour with unitary operators in this lecture.

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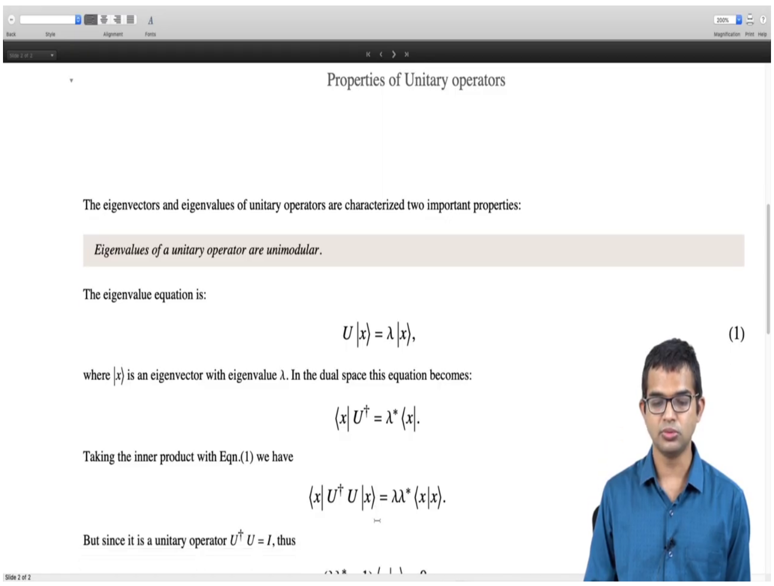

So, the first property is that eigenvalues of a unitary operator are unimodular, right. You cannot say anything about the reality or not, there then forget to be. They may be real, but it is very rare in fact, but they are going to be unimodular. What does that mean? We will explain here.

So, you have U acting on x giving you some lambda times x, right. So, lambda is an eigenvalue, U is a unitary operator. Now, in the dual space, this means the bra x acting which is being acted upon by U dagger is equal to lambda star x the bra vector x. So, if you take the inner product you know with equation 1, you have x U dagger U x is equal to lambda lambda star x x.

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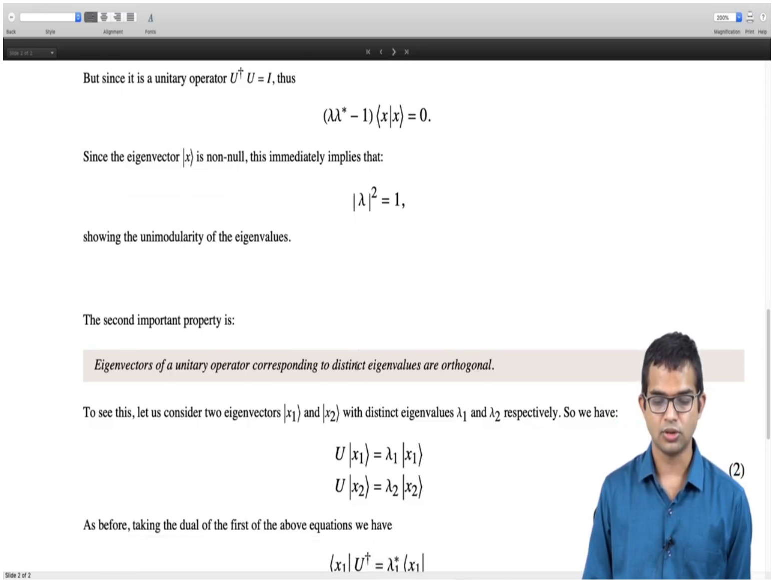

But U dagger U is the same as identity, so that means, the inner product of x with x is equal to lambda lambda star inner product of x with x. So, lambda lambda star minus 1 times the inner product of x with x equal to 0.

Now, x is a non null vector otherwise it cannot be an Eigenvector and therefore, lambda lambda star must be equal to 1, right. So, that means, that mod of lambda squared is equal to 1, right. So, this is what is meant by unimodular. It simply means that lambda lambda star must be equal to 1, right. I mean it is possible for this to be real, but that is you know that is a very rare scenario.

So, in fact, you can write lambda as e to the I theta. If theta also is 0, right or pi that is when you can get a real eigenvalue as well ok. So, the key point is that the Eigenvalues of unitary operators are unimodular.

Now, let us look at what happens to Eigenvectors corresponding to distinct Eigenvalues. Again Eigenvectors of a unitary operator corresponding to distinct eigenvalues are going to be orthogonal, right. How do we see this? So, let us consider two eigen vectors x 1 and x 2 distinct eigenvalues lambda 1 lambda 2, right. U x 1 gives you lambda 1 x 1 and U x 2 gives you lambda 2 x 2.

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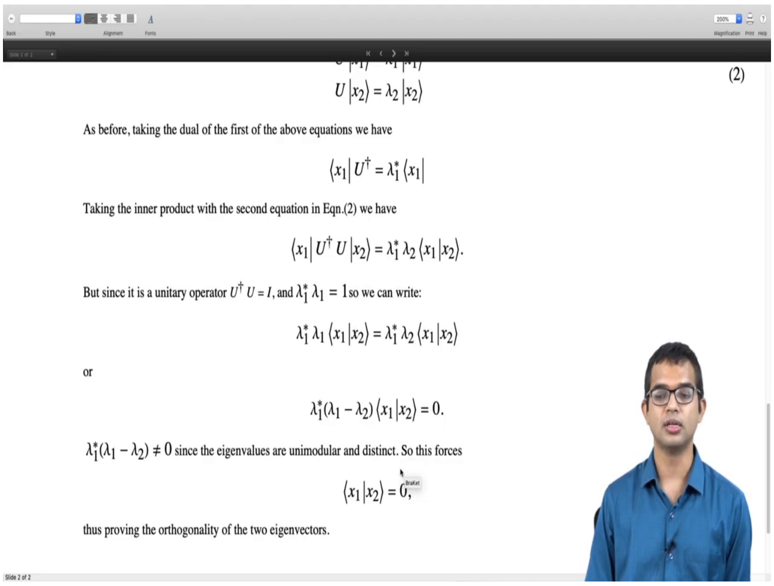

So, taking the dual we have of the first of these two equations, you have x 1 U dagger is equal to lambda 1 star x 1. Now, taking the inner product with the second equation, in equation 2 we have x 1 U dagger U x 2 is equal to lambda 1 star lambda 2 x 1 x 2. But U dagger U is equal to the identity. So, you have and ah yeah; so, and we can also put in a lambda 1 star lambda 1 for free on the left hand side because lambda 1 star lambda 1 is equal to 1.

So, we will write down the left hand side as x 1 x 2 and in place of 1, we will just put lambda 1 star lambda 1 is equal to lambda 1 star lambda 2 x 1 x 2 on the right hand side as it is. So, rearranging terms we have a lambda 1 star into lambda 1 minus lambda 2 into x 1 x 2 equal to 0.

Now, lambda 1 star is unimodular, right. If lambda 1 is unimodular, so is lambda 1 star. So, there is no question of lambda 1 star being 0. And nor is it possible for lambda 1 minus lambda 2 to be 0; because they are distinct eigenvalues, therefore, that gives us the only possibility which is the inner product of x 1 with x 2 0, forcing these two eigen vectors to be orthogonal.

So, once again the eigenvectors of unitary operators all span the space, right, right which is a consequence of this. But at this point we have just managed to show these two facts. First is that unitary operators have unimodular eigenvalues and eigenvectors corresponding to distinct eigenvalues of unitary operators are also orthogonal. That is all for this lecture.

Thank you.