Mathematical Methods 1
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Linear Algebra
Lecture – 28
Hermitian operators

Ok. So, we have seen what operators are, what linear operators are, we discussed the crucial concept of eigenvalues and eigenvectors corresponding to linear vectors in an arbitrary linear vector space.

So, now in this lecture we will define an important class of linear operators namely Hermitian operators. And also look at some consequences for the eigenvalues and eigenvectors of Hermitian operators, you know just from their property of Hermiticity which we will describe in this lecture, ok.

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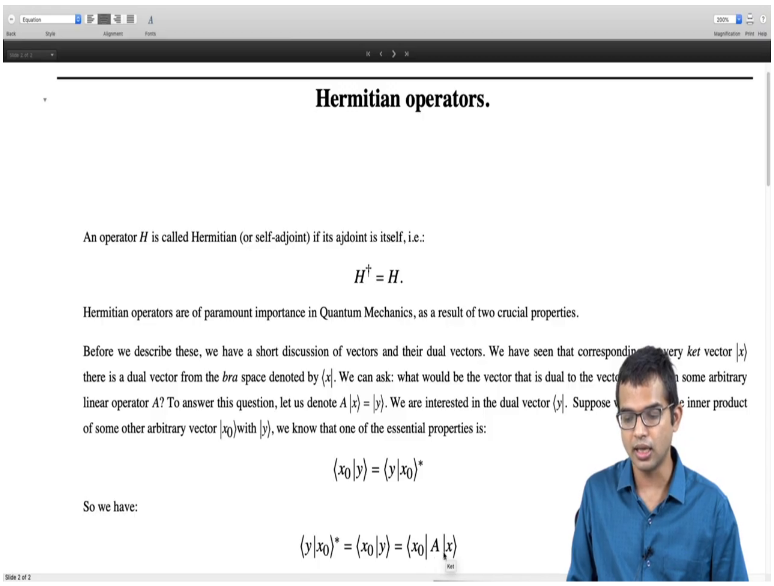

An operator is called Hermitian or self-adjoint if its self-adjoint meaning the adjoint operator corresponding to this operator is itself. So, H dagger is equal to H, right. Hermitian operators are of great importance in quantum mechanics and as a result of two crucial properties that we will discuss in this lecture, right.

So, in quantum mechanics we know that all physical observables, you know there are operators corresponding to all physical observables and these operators have to be Hermitian, right. And you know one reason for this is the two properties that we will describe in this lecture.

And before we go to these properties you know let us ask let us you know point out an important fact, right based on what we have already defined, right. So, we have said that every ket operator corresponds to a bra operator, every ket vector, right. There is corresponding to this there is a dual vector in the dual space which is called a bra vector and which is denoted by this symbol.

So, we can ask, suppose we take some linear operator and act on some vector x which is a ket vector. Then we know that A acting on x will give you another vector which is also another ket vector. So, what would be the vector which is dual to this? What is the vector in the dual space? Right. So, clearly you would guess that there would be some relationship to the bra vector x. But, what is the precise vector?

So, to answer this question let us define A acting on x to be some other ket vector y, right. We are interested in the dual vector; you know this is bra y. Now, suppose we consider the inner product of some other arbitrary vector x naught with y, we know that one of the essential properties is inner product of x naught with y is the same as the inner product of y with x naught with a complex conjugate, right on it So, we have but y is; so y x naught star is equal to x naught y, but y is A acting on x, so x naught A x, right.

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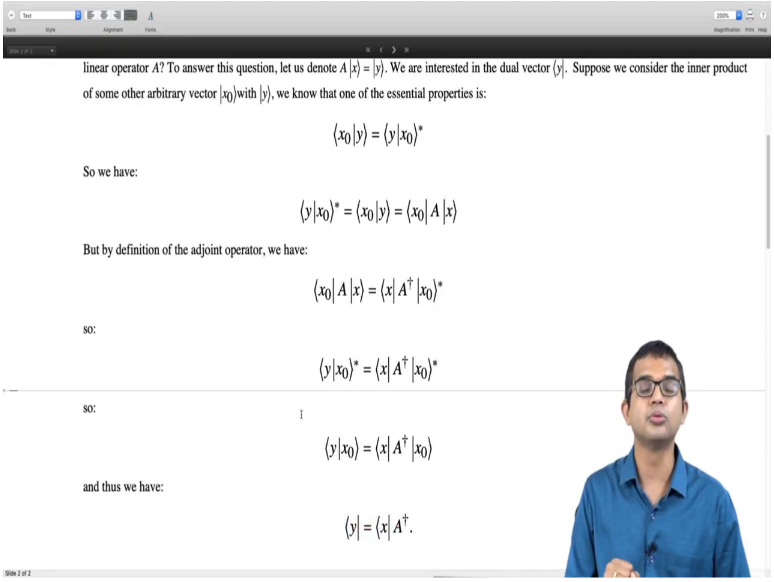

But by definition of the adjoint operator we have you know this matrix element x naught A x is the same as x A dagger x naught the whole thing complex conjugate. This is the definition of the adjoint operator using matrix elements, right. So, we have y x naught star is the same as x naught A x which is the same as x A dagger x naught star. So, removing the complex conjugates on both sides we have y x naught is equal to x A dagger x naught, right.

And thus, we have you know this stuff you can look at this stuff you have, you know this is this is a bra space vector, and if so have our bra space vector that you are going to you are taking the inner product of this with the same vector x naught and this holds for any arbitrary x naught.

Therefore, this bra space vector y is the same as you know this bra x with A dagger right. So, this is the vector that we are interested in, right. So, whenever you have you know x the ket x corresponds to the bra x and A x corresponds to x A dagger, right. So, this is an important fact, right.

Now, and this is intimately connected to you know this anti-linear nature of this inner product operation, right when you we have seen that if you have you know alpha times x plus beta times y and if you take the dual vector corresponding to alpha x plus beta y it will, it is going to become alpha star and beta star, right.

So, if you consider the inner product operation - if the ket vector has some alphas and betas involved they will just come out. But if those linear combinations are in this bra space that is going to give you alpha stars and beta stars, right.

So, this A dagger also corresponds to this you know this anti linear nature of the inner product operation, right. So, they are all very closely related. But anyway, this is a fact that we will have to use in all kinds of manipulations when we are working with ket vectors and bra vectors. So, it is you might as well you know spend some time and understand it well, ok.

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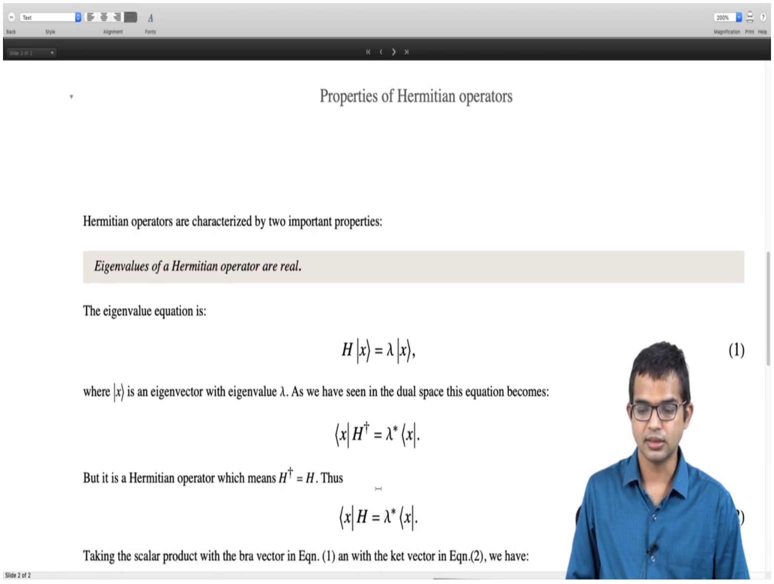

So, now let us go to our two crucial properties of Hermitian operators. Property number one is that eigenvalues of Hermitian operators are real, right. So, how does this come about? So, the eigenvalue equation is H acting on x is equal to lambda x. So, you are given some Eigen, we have managed to find some eigenvector with x and its eigenvalues lambda x.

So, now let us take the dual space vector of this. So, we are just saying that you should take the dual space vector. It is going to be bra x H dagger and is equal to lambda star, you have to do this, right.

So, this is the anti-linear you know property when you take you know any constant is associated, if you multiply a ket vector with a constant a bra vector will be multiplied with the constant the complex conjugate of that constant, right. So, lambda star x. Now, but H dagger is the same as H that is the defining property of the Hermitian operator, so you have x H is equal to lambda star x.

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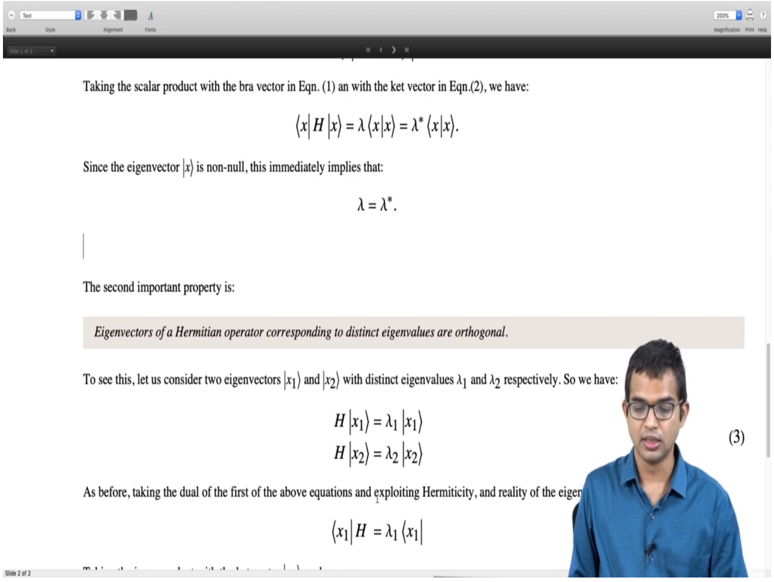

Now, taking the scalar product with the bra vector in equation 1, right. So, and with the ket vector in equation 2, we have x H x is equal to lambda. So, if I bring in x from the left hand side in equation 1, right, then I get x H x is equal to lambda x x, but I could have also brought a ket x here from the right side here. So, then it will give me lambda star x x, so which is equal to lambda star x x both of these are equal to you know this matrix element x H x which comes from the Hermiticity of this operator H.

Since, the eigenvector x is non-null this immediately implies that lambda is equal to lambda star, right, there is no other way, right. If you have the equivalence of these two and x is non-null, that means lambda must be equal to lambda star which implies that lambda is real, right. That was the first property, right.

And this is a property that is essential for physical observables, right. Eigenvalues of Hermitian operators correspond to observables, the eigenvalues are really where the physical information is and that has to be real. That is why this theorem is of great importance in quantum mechanics as you might have seen already.

The second property is that eigenvectors of a Hermitian operator corresponding to distinct eigenvalues are orthogonal, right. So, let us see how this comes about. So, suppose you have two eigenvectors x 1 and x 2, and distinct eigenvalues lambda 1 and lambda 2, respectively.

So, we have these two equations H acting on x 1 is equal to lambda 1, x 1 and H acting on x 2 is equal to lambda 2 x 2. Now, if we take the dual of you know the first equation x 1 H lambda 1 x 1. I have already used the fact that H dagger is equal to H and lambda 1 star must be equal to lambda 1 because it is a Hermitian operator.

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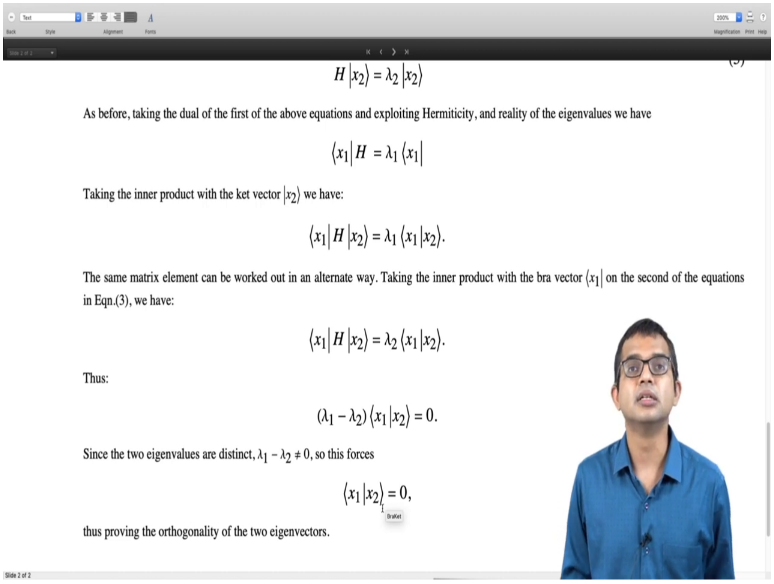

Now, if I take the inner product with this vector x 2. Here I have x 1 H x 2 is equal to lambda 1 x 1 x 2. But I could have calculated the same matrix element in an alternate way. I can go to equation 3, and then multiply or you know take the inner product from the left side with x 1, here in the second equation, then I have x 1 H x 2 is equal to lambda 2 x 1 x 2, right.

So, I have for the same matrix element, but two different values computed here lambda 1 x 1 x 2 or lambda 2 x 1 x 2, both of them are the same. So, lambda 1 minus lambda 2 times x 1 x 2 equal to 0. And we have been told that lambda 1 is not equal to lambda 2, right, it is part of the proposition of this statement. So, there are distinct eigenvalues. So, x 1 the inner product of x 1 and x 2 is equal to 0, proving orthogonality of the two eigenvectors, right.

So, a consequence of this is that the eigenvectors of a Hermitian operator you know are complete if they are able to span the whole space and this is of paramount importance in quantum mechanics, right. So, it means that you should be able to write any state as a linear combination of Eigenfunctions of any observable, right. And so which means first of all that H becomes diagonalizable and it is a complete basis the Eigenfunctions provide and so on, and of great importance, right.

So, here we have only shown that any two eigenvectors corresponding to distinct eigenvalues are orthogonal, but from this there is a consequence that is completeness, right. So, we will return to this later on when we look at matrices, right. But at the moment we are looking at abstract operators and this is a result that can be shown for abstract operators. So, that is all for this lecture.

Thank you.