Mathematical Methods 1
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Linear Algebra
Lecture – 27
Eigenvalues and Eigenvectors

Ok. So, we have looked at what operators are on an abstract linear vector space; operators take vectors and give you other vectors and then, we have looked at what linear operators are right. So, they obey the principle of superposition. Now, we will discuss an aspect which is central to the set of topics that we have been covering; namely, eigenvalues and eigenvectors right.

So, perhaps you might be familiar with this notion in the context of matrices; but we will define it you know from the point of view of an abstract linear vector right and then, we will see how this is connected to eigenvalues and eigenvectors of matrices later on during the course of our discussion ok.

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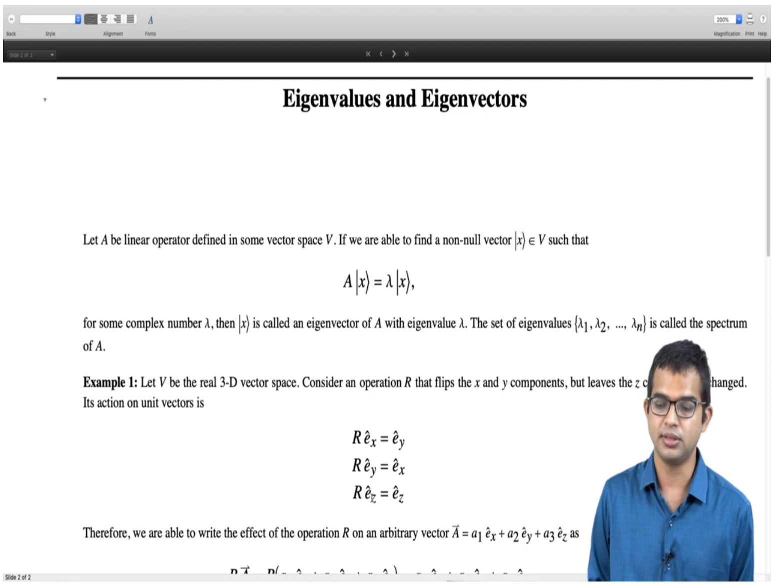

So, let A be some linear operator which is defined on some vector space V. If we are able to find some non null vector x such that if you operate with A on x, it must give you back the same vector itself, you know subject to some coefficient which is called the Eigenvalue right. If you are able to find such a vector, then it is called Eigenvector and you know lambda could in general be some complex number and the lambda would be called an eigenvalue corresponding to that eigenvector x.

Now, if you find all Eigenvalues corresponding to some operator A and you call it the spectrum of that operator A right. So, the Eigenvector right is you know subject to some factor. So, you can clearly see that if you take if x is an eigenvector, so is alpha times x for any arbitrary complex number right. You can put alpha on the left hand side and on the right hand side, nothing changes; Eigenvalue remains lambda.

So, typically, we tend to look at all these alpha x's as really the same right. So, you do not make a distinction between x and alpha x. If x is an eigenvector, then so is alpha x that is not a distinct eigenvector right. But given you know there are cases, where you can have for the same eigenvalue multiple linearly independent Eigenvectors. But you know we are getting ahead of ourselves.

So, I am just telling you at this point that if you have an eigenvector x, then so is alpha times x and that it is kind of trivial - a lot of different vectors being an eigenvector. But you tend to think over them as the same, really. So, let us look at a few examples right. So, at this point, we are still thinking of these vectors as abstract vectors and also, these operators are as you know abstract operators.

So, let us consider some you know the you know let me be the real 3-D vector space and let me think of this operation R that flips the x and y components; but leaves the z component unchanged. So, its action on you know all the vectors can be entirely different by just defining its operation on the unit vector. So, let us look at e x, e y and e z. So, it will operate on e x and give you e y or it operates on e y and will give you e x and when it operates on e z, just returns e z.

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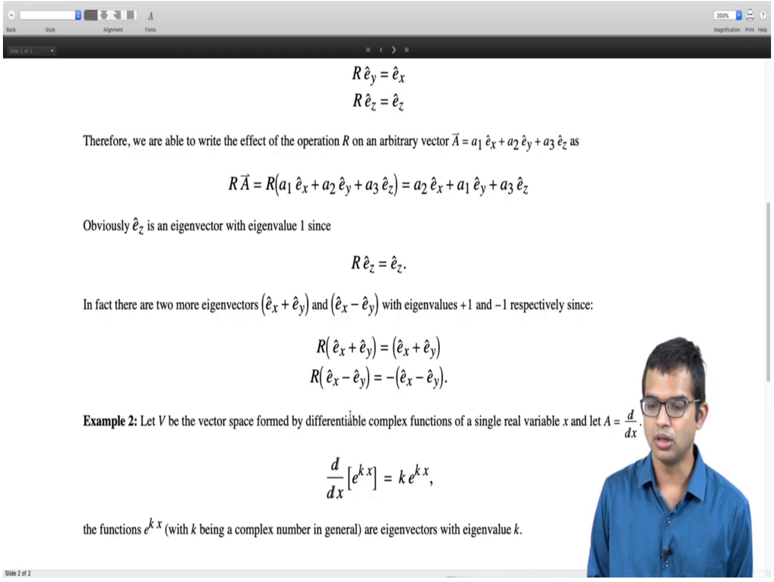

So, in general if it acts on some vector A is equal to a 1 e x plus a 2 e y plus a 3 e z, then you would get a 2 e x plus a 1 e y plus a 3 e z right. So, it is obvious that e z is an eigenvector right. From the definition itself, you see that R acting on e z gives you e z. So, nothing changes. So, therefore, its e z is one eigenvector with eigenvalue 1; but there are two more eigenvectors for this linear operator right.

You know you might be able to quickly look at the structure and guess what this is indeed. So, e x plus e y and e x minus e y are Eigenvectors. As you can check, one of them has eigenvalue 1 and the other one has Eigen value minus 1 because R acting on e x plus e y will just give you ex plus e y; but R acting on e x minus e y will give you minus ex minus e y right.

So, this is one example, but let us look at another example, where we consider an abstract linear vector space made up of differentiable complex functions of a single variable x and let us consider d by dx as the operator right. And then, we are looking for you know eigenvectors.

So, d by dx acting upon some vector must give you the same vector times some constant. So, you can check that if you consider the function e to the k x, we will get k times e to the k x. So, k can be some arbitrary complex number in general and it will give Eigenvalue k right.

I have just considered two examples to illustrate that your operators can be very abstract in nature and can come from many different formulations of a linear vector space and still this notion of eigenvalue and eigenvector is defined right.

So, later on, we will see that you know considering matrices and their Eigenvectors and eigenvalues like most of us are already familiar with, you know there is a close connection between this and that particularly in the context of finite dimensional linear vector spaces or that is coming a bit later.

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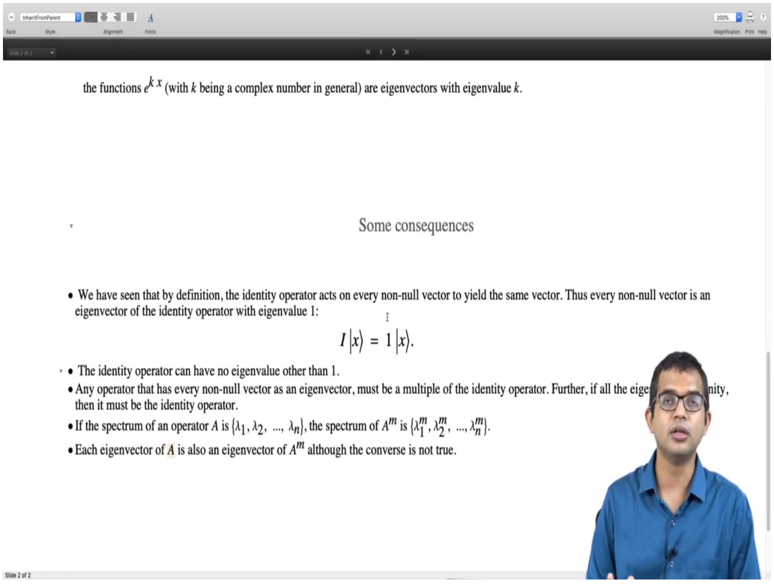

So, let us look at some immediate consequences of this definition right. One pertains to the identity operator right. So, the identity operator whenever it acts on any non-null vector, it must return the same vector. So, that is really the same as the definition of this Eigenvalue with.

See it says, what it says is the identity operator is you know every non-null vector is an eigenvector of the identity operator with eigenvalue 1 and so right and also, there is a consequence of the definition is that the identity operator can have no other eigenvalue.

Because every non-null vector will give you back the same vector. So, when you operate with I. So, you have only one as a eigenvalue and there is no other Eigenvalue and any operator that has every non-null vector is an eigenvector must be the identity operator or a multiple operator right and if you say that eigenvalues are unity, then it is a identity operator right.

So, I mean we saw that the identity operator has this decomposition. You could write it as e 1 e 1 plus e 2 e 2 so on in some basis and then, we argued that you know I acting the way, we showed that this operator in this outer product form right is the identity operator few lectures ago was to show that it whenever it operates on any basis vector, it give you it returns the same value same vector and therefore, it would give you the same vector for any arbitrary vector right.

So, these are all very direct consequences of the definition of the identity operator. So, also it follows from this definition that if the spectrum of some operator A is lambda 1, lambda 2 all the way up to lambda n, the spectrum of the mth power A to the m is going to be a lambda 1 to m lambda 2 to the m so on as you can verify right.

You operate with A on x gives you lambda x, if you again bring another A from the left side, you have A squared acting on lambda on x is equal to lambda squared acting on x and then, lambda cube so on right. Every time you bring in another factor A, that will give you an extra factor of lambda.

So, each eigenvector of A is also an eigenvector of A to the m. So, but the converse is not true right. You can find an example right. So, I mean one way to immediately see this is if A if you have an eigenvector of x. So, you have A x is equal to lambda x. So, A to the power m, you know every time you will keep on getting, you will get back the same vector, if you no matter how many A's you put in right.

But if you have found one Eigenvector for A to the m, it doesn't mean that it is going to be an eigenvector of A. So, the simplest way to you know test this is to come up with an example, which you can do. If you consider a you know think of a 2 by 2 matrix it is the simplest way right. You can think of more complicated abstract operators. But think of a 2 by 2 matrix and show you know just one counterexample whether A squared has a certain eigenvector, but it is not going to be an eigenvector of A.

Maybe this can be a homework problem. But this is a quick introduction to the notion of eigenvalues and eigenvectors for linear operators on linear vector space.

Thank you.