Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Linear Algebra
Lecture – 26
Projection operators

Ok. So, we have seen how the identity operator has a convenient representation in terms of the summation of these e i e i operators of some orthonormal basis, right. So, we will look at a similar expression for what are called Projection operators in this lecture and you know connected to the notion of the direct sum of subspaces ok.

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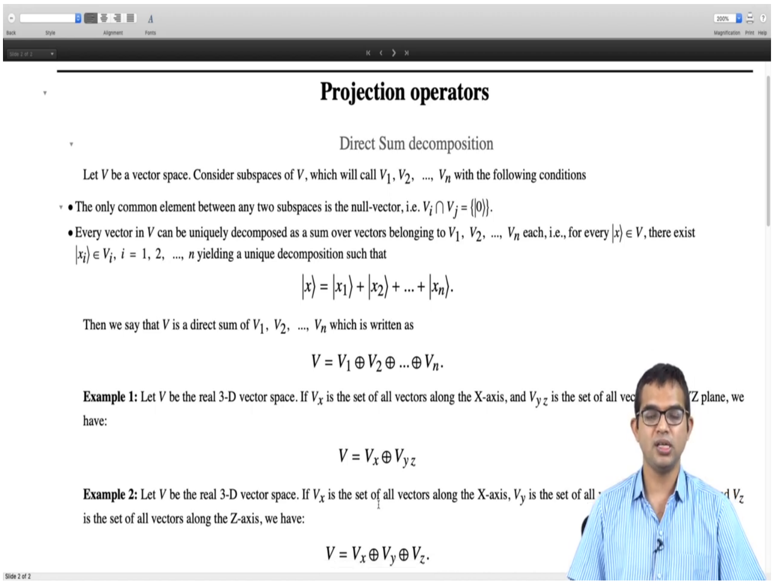

So, let V be some vector space and we can think of a bunch of subspaces you know V 1, V 2, and so on with the following conditions, right. So, any two subspaces V i and V j have only a single element precisely one element which is common to them right that is the; that is the null vector, right.

So, since all of them are subspaces all of them must have you know the null vector and so that is the only common element. So, suppose we construct a bunch of subspaces in this manner and every vector in V can be uniquely decomposed as a sum over vectors belonging to V 1, V 2 so on all the way up to V n, right you know.

So, that is, for example, if you have a vector x then there you will be able to find vectors x i, where x i belongs to V i all the way i going all the way from 1 to n and which yields a unique decomposition x can be written as x 1 plus x 2 plus x 3 all the way up to x n. I think this is the way you have constructed your subspaces and it is possible to do. So, I will give you examples.

Then we say that V the overall space is a direct sum of V 1, V 2 so on all the way up to V n which is written as V 1 plus this you know this symbol V 2 so on all the way up to V n, right. So, one immediate example which comes to mind is that of 3D vector space right. So, 3D vector space you know has all kinds of vectors, but there is a subclass of this which is the set of vectors which all live on the X-axis which point along the x X-axis.

Now, there is another subspace V iz which corresponds to all vectors which live in the YZ plane right. So, the only vector which is common to these two subspaces is the origin or the null vector right. So, the null vector is the only vector which is common to both of these spaces and any vector in general can be written as the sum of you know vectors drawn from these two spaces that exactly like the requirements.

So, therefore, we can write V is equal to V x direct sum V yz, but V yz itself you know by the same you know by a similar procedure we can break the V yz itself into V y and V z. We can introduce the set V y as a set of all vectors which lie along the Y-axis and V z as a set of all vectors which lie along the Z axis. So, in fact, we can write this overall space V as the direct sum of V x, V y and V z, right.

So, this is the idea of direct sum decomposition right of a bigger space in terms of many subspaces and there is a very tight way of doing this such that there is no redundancy right between the subspaces.

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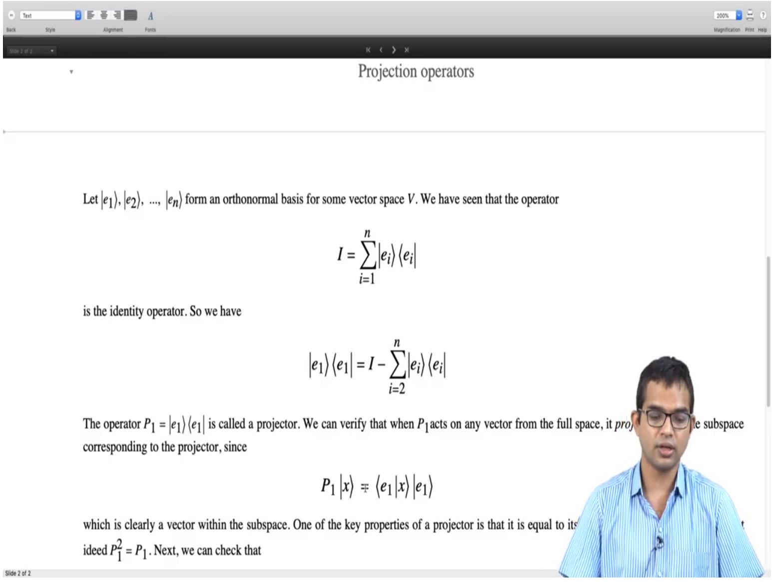

Now, let us look at the idea of projection operators, right. So, we have seen that if you have some orthonormal basis for the overall space V right and subspaces have their own orthonormal basis right, but let us say that we are thinking of this orthonormal basis e1, e2, all the way up to e n for the vector space V.

Now, we have seen that if you consider this operator summation over i e i e; i is the identity operator, right. Now, if we remove if we do not keep all of these right if or if you keep only one of these let us say. So, if we consider this operator e 1 e 1 then that is nothing, but i minus summation over i going from 2 to n e i e i.

Now, this operator is an example of what is called a projection operator - think of what this operator does to see to get a feeling for what it is, right. It is if you take this operator and act upon any vector of your full space it will necessarily give you a vector within the subspace only, right.

So, we can see that P 1 acting on x will give the inner product of e 1 x. So, the component of it will pull out the component of that vector along the only direction that is represented; e 1 is the only unit vector. So, it will pull out the component of that vector in that space and give you and its direction is e 1 now, right. I mean you can think of this as representing a direction e 1.

Now, it is clearly a vector within the subspace right. One of the essential properties of an operator to be a projector operator is something we can verify for P 1 which is that P 1 squared is equal to P 1 itself right. So, think of what P 1 squared is doing, right. So, P 1 will take any vector in general and project it onto the space itself.

Now, if you try to project this vector which has already been projected into the space a second time, that is what P 1 squared is doing, then there should be no change, right. So, P 1 squared is the same as P 1. So, there is no need to project a vector which has already been projected. So, that is the physical content of this statement P 1 squared equal P 1. So, this is in fact, a way to check whether an operator is a projector or not it is to verify if P 1 squared is equal to P 1.

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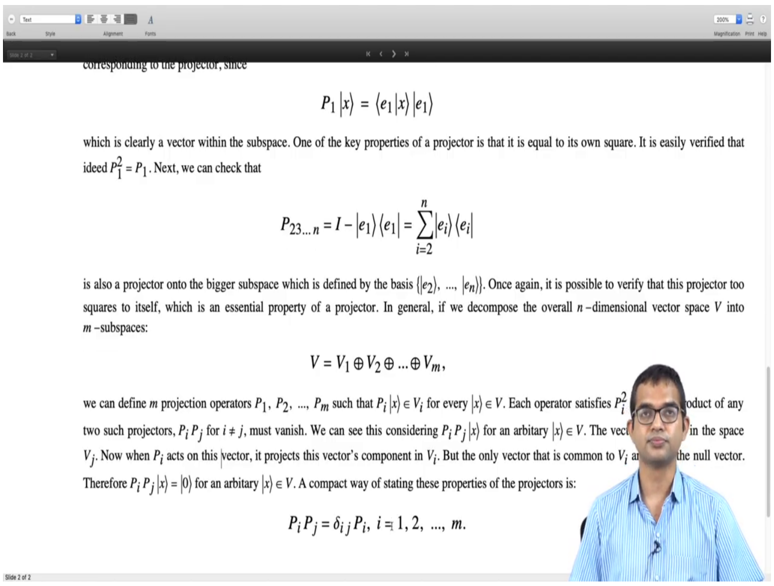

Now, next we can check that you know I said e 1 e 1 is a projection operator, but in fact, i minus e 1 e 1 is also a projection operator, you can check this, right? 2, 3 we can think of this as a bigger space. It is projecting onto not a space with just one you know basis element it is not a dimension 1 space, but it is a dimension n minus 1 space, right.

So, the dimension of your subspace can be small, or the dimension can be big, but there are projection operators which you know collect together you know some basis elements of that subspace and you know if you create an operator of this kind belonging to some space that becomes a projector.

Now, this is something that you should verify. You should take this operator you know i minus e 1 e 1 squared and check that indeed you get back the same operator right. So, this is something for you to do as exercise right. So, since that is an essential requirement for this to be a projector. Also you can try to operate with this on any state and show that it will always lie in this subspace ok.

So, in general we can decompose the overall n-dimensional vector space V into m subspaces, right. It may be equal to this m may be equal n or it may be some other number, but you can break down your overall space into many subspaces, right. We have seen that there is this direct sum decomposition.

So, suppose we do this we can find you know projectors corresponding to each of these subspaces. So, in fact, we can find any basis in V 1, any basis for V 2, any basis for you know each of the V i's all the way up to V n and you know use something like this type of an operation within the basis of that subspace and you will get a projector for that subspace.

Now, these projectors are going to project any vectors from the overall space into the subspace V i acting on x will be an element of V i for every x is an element of V and each of these operators must satisfy the condition P i squared equal to P i, right. If it does not then it is not a projector.

The product of any two such projectors is also interesting and it is going to vanish because you know of the nature in which you have divided these subspaces, right. There is a direct sum decomposition. So, the overlap between any two subspaces is you know there is just the null vector which is the only common element between any two subspaces.

So, if you consider such a decomposition and if you construct these projection operators then P i, P j is going to vanish unless i is equal to j right. So, we can see this you know from considering P i, P j what this product does to an operator x to a vector x and arbitrary vector x which is an element of V. So, P j x right because that is the property of a projection operator is going to bring it to a vector which lies in the space V j.

Now, if you take P i and act upon this vector which lies in V j, then this since it lies in V j which is you know which is an orthogonal space to P i. So, it is not going to have any component in V i right now P i is going to try to project these vector components in V i, but the only common vector between V i and V j is the null vector.

So, in fact, all that P i is going to manage when it acts upon any vector from V j is to give you the null vector and that is what happens. Therefore, P i, P j acting on x is going to give you the null vector for an arbitrary vector x is an element of V and therefore, P i, P j itself is the null operator.

So, a compact way of stating these properties is to say you know P i P j is equal to delta ij P i, right. When P i when i is equal to j P i squared is going to P i write if i is not equal to j, then P i times P i is the null operator ok. That is all for this lecture.

Thank you.