Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Linear Algebra
Lecture - 20
Span, basis and dimension of a LVS

So, after our detour involving a discussion about matrices, the rank of a matrix, you know determinants and properties and so on, now we return to our abstract linear vector space and we define some very important properties of a linear vector space. So, there is a notion of a Span, the notion of a basis and dimension of a linear vector space, right. So, these are the topics which will be covered in this lecture, ok.

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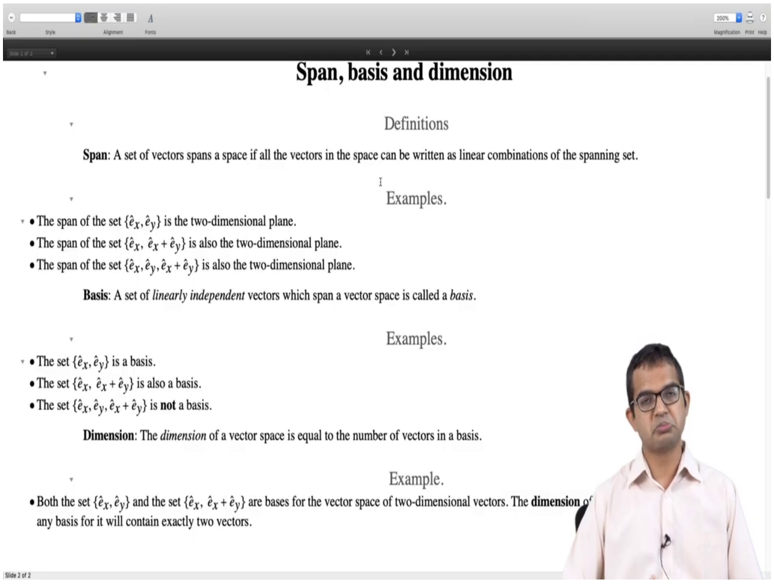

So, if you are given a bunch of vectors, so the span of you know the set of vectors is simply all possible vectors that can be produced as a linear combination of these vectors, right. So, you say that a set of vectors spans or space, if all the vectors in the space can be written as linear combinations of the spanning set or you might also think of this the span of a given set of vectors is a space, right.

You say that a set of vectors spans or space, if every vector in that space can be written as the linear combination of a set of vectors and alternately the span of a set of vector is the space which is created by you know coming with up with all linear combinations of vectors drawn from that set, right. So, that is the notion of a span.

Now, for example, right if you take the set e x and e y two-dimensional vector, so the unit vector along the x direction the unit vector along the y direction, the span of just these two vectors is actually the infinite two-dimensional plane. So, every vector in 2D can be written as you know a x times e x plus a y times e y. So, likewise the span of the set e x and e x plus y e y is also the two-dimensional plane, right.

So, these two are not orthogonal, but they are the just these two vectors are enough to generate for you the whole two-dimensional plane. And likewise, in fact you may put in more vectors drawn from the two-dimensional plane. For example, you could consider e x, e y and e x plus e y, but every vector that is obtained as a linear combination of these 3 vectors is still going to be just the two-dimensional plane; you are not going to get any vector which lies outside of the 2D plane.

So in fact you can put more vectors into this set; as long as they are all you know vectors within the two-dimensional plane, the span of this set that you are going to come up with is still going to be the two-dimensional plane, right.

So, you see that there is you know on the one hand there is a notion of a span of a set and then you might start thinking is there some minimal set, right, it seems like there is a lot of redundancy in these kinds of you know sets that you may come up with, right. So, that is where the notion of the basis comes in, right.

So, the minimal set which can generate for you the whole space is what is the idea of a basis, right. So, here we have to bring in not only the idea of a span, but also that of linear independence, right. So, if you also impose linear independence on these vectors, you have a bunch of vectors which are all linearly independent and then if you take the span of such a set and you create a space, right. So, this set is going to form a basis for this space.

So, another way of thinking of this is you have a base you have a space already a vector space is given. Now, a basis for this vector space is any set of vectors which are all you know which have to satisfy two properties; one is they have to be mutually linearly independent, right, and they have to span the space. So, which means basically that every vector in that space should be presentable as a linear combination of the vectors from this set that you have taken and then it becomes a basis, right.

So, an example here is e x and e y you know we again we look at the same 3 examples we considered with regard to the span e x and e y that is going to be a basis; e x and e x plus e y is also a basis, right, although they are not orthogonal to each other.

So, I mean e x and e y we have been using e x and e y as a basis all along, right without perhaps thinking of them as a basis because it is a very convenient basis to work with. e x and e x plus e y perhaps is not such a convenient basis, but it is still a basis all right, right, because any vector in the two-dimensional space can be written as a linear combination of e x and e x plus e y, right. This is something that you can check, right.

In fact, we did this; we already showed this for arbitrary vectors. If you have any two arbitrary vectors a and b, right in some plane, any other vector can be written as a linear combination of a and b. We showed this you know using the cross product of these vectors, which uniquely defines the direction perpendicular to the plane therefore, the plane is defined and so any other vector can be obtained in terms of a and b, right.

So, here e x and e x plus e y for sure they will form a basis, but the set e x, e y and e x plus e y is not a basis, right. The reason is there is a redundancy here, right. So, the basis in some sense is a very compact set, right. It does not have any more information than necessary and it cannot have you know less information than necessary. It needs to have exactly you know the relevant amount of information, right.

So, what is that relevant? How much is that relevant information? There is something which seems to be conserved, right. You can have many different kinds of basis, but apparently all of them are able to generate for you the whole space in a very, so that there is also no wastage and there are no unnecessary vectors here. So, what is this thing that is getting conserved? And that is the idea of the dimension of a vector space.

So, the dimension of a vector space is a well-defined object, right. It is just equal to the number of vectors in any basis, right. As you have seen you can think of many different ways. I could have thought of it as e x and e x plus e y or I could have created a basis made up of e y and e x plus e y or e y and you know three times e x plus four times e y. You know I can think of many different ways of constructing a basis, right.

But one thing for sure is if you are looking at this 2D space or any basis is going to have exactly two elements, right and so that is the idea of the dimension of a vector space, right. So, there you go.

So, once again if you look at these examples e x and e y and e x and e x plus e y or you can think of e x plus e y and e x minus e y, you know all of these are basis, you can construct basis in you know zillions of different ways. All of them will have exactly two elements as far as this vector space of two-dimensional vectors is concerned. So, the dimension of this vector space is 2.

Now, in all our discussions we. So, you see that although the dimension of your space is just 2, right, it is finite and in fact it is a very small number it is just 2 and yet the number of vectors in the space is actually infinite, right. You can use a relatively small number of vectors and create infinitely many vectors, right. So, although the number of vectors in your set is an infinite set, it has a finite dimension, right.

So, in our discussion we are going to restrict ourselves to finite dimensional vector spaces. So, of course, you know as you might guess there are also infinite dimensional vector spaces and so there you know one has to be more careful, right.

So, with regard to you know some of the results that we prove for example, right if you take all complex numbers we have seen that it is a vector space, you can if you add any two complex numbers you get a you are going to get another complex number and so on, right.

So, you can think of many infinite dimensional vector spaces as examples. But for all the results that we are going to describe - we will stick to finite dimensional vector spaces, ok.

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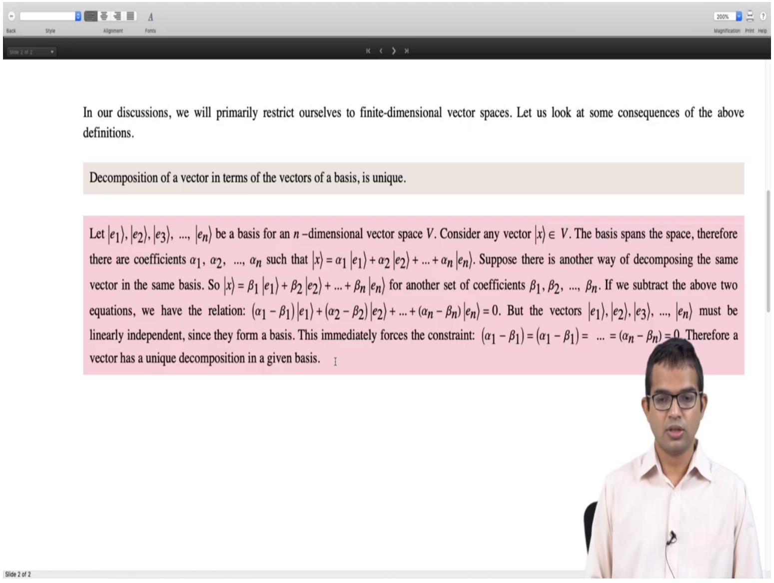

So, let us look at some important consequences of just you know this set of definitions that we have made, span basis and dimension, right. So, one is the decomposition of; so, we have said that for a set to be a basis, it must be able to represent any vector in that space as a linear combination of those vectors from that basis.

Now, we are already able to show that the decomposition of any vector in terms of the vectors of a basis not only exists, but it is unique. That it exists already has been said within the definition, but now we can show that it is unique. And this in fact follows from the linear independence of the vector.

So, let us look at this argument. If you want to pause the video and try to work out the argument for yourself please feel free to do so. So, here is the argument. So, let e 1, e 2, e 3, so on, e n be a basis for an n-dimensional vector space V. Now, consider any vector x which is an element of this space.

Now, the basis must span the space, right otherwise you would not call it a basis. Therefore, you will be able to find some coefficients alpha 1, alpha 2, alpha n, the scalar such that you will be able to write, you know expand your vector x as alpha 1 e 1 plus alpha 2 e 2 so on plus alpha n e n.

Now, but suppose this is not unique, suppose you know your friend tells you that you have another he has found another way of doing this expansion. So, there is another set of coefficients beta 1, beta 2, all the way up to beta n x you know where you write x as beta 1 e 1 plus beta 2 e 2 all the way up to summation beta n en. Now, we will show that this immediately implies that alpha 1 is equal to beta 1, alpha 2 equal to beta 2 and so on all the way up to alpha n is equal to beta n which means that there is a unique decomposition.

So, how does this come about? So, we just simply subtract these two. You have an equation for x and you have another equation for x, so if you subtract these two you are going to get 0, the 0 vector which you know you can think of as a 0 as well, right. So, on the right hand side you have the 0 vector and on the left hand side you have alpha 1 minus beta 1 times e 1 plus alpha 2 minus beta 2 times e 2 so on all the way up to alpha n minus b n e n.

So, these vectors e 1, e 2, e 3, and so on must be linearly independent, right. So, if they must be linearly independent, the definition of linear independence is that you will never be able to find some set of coefficients which are not all 0; such that you know the set of coefficients times the various elements of, the various vectors involved here summation over all of them is equal to 0, right. That is precisely what we have here.

We seem to have found a set of coefficients alpha 1 minus beta 1, alpha 2 minus beta 2 so on all the way up to alpha n minus beta n such that if we type these coefficients along with them with the vectors and add them we are getting 0, right. If these vectors have to be linearly independent it immediately implies that each of these coefficients that is alpha 1 minus beta 1 is equal to 0, alpha 2 minus beta 2 equal to 0 and all the way up to alpha n minus beta n is equal to 0.

Therefore, the decomposition of this vector x must be unique, right. So, this follows directly from the linear independence of the vectors of a basis, right.

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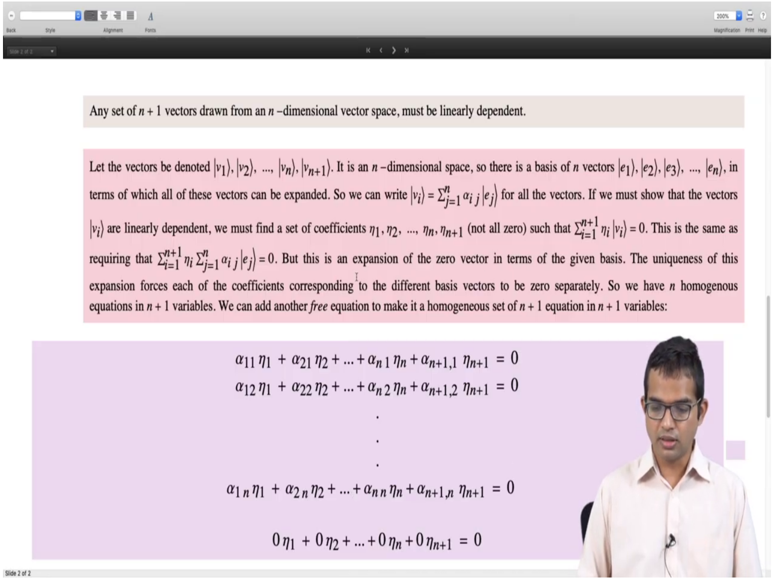

So, next we have another result. This is a very beautiful result, right. So, any set of n plus 1 vectors drawn from an n-dimensional vector space must be linearly dependent, right. So, if you want to pause the video and try to work this out on your own please do so. So, this again we will use some of the ideas from matrices that we worked on you know some few lectures ago. So, let us look at the argument now.

So, you have a set of vectors v 1, v 2, v n and also v n plus 1, you have given n plus 1 vectors. So, the claim is that any set of n plus 1 vectors in an n-dimensional space vector space necessarily must be linearly dependent, right. It is an n-dimensional space, so there is a basis of n vectors, e 1, e 2 all the way up to e n these are all linearly independent and which can span the space.

So, we should be able to write v 1 as you know expand v 1 in terms of all these n vectors from the basis, v 2 also in terms of the basis, v 3 so on all the way up to v n plus 1. So, let us do that. And we have this matrix of coefficients alpha ij, right. I am writing v i as summation over j alpha i j e j, right.

Now, if we must show that these vectors v i are linearly dependent, so that means, we just need to find a set of coefficients eta 1, eta 2, all the way up to eta n plus 1, right all of them nonzero, right. You cannot, if you have all of them 0 that is a trivial statement, right. You should be able to find a set of coefficients such that summation over i, eta i v i equal to 0, right that is the condition for linear dependence.

So, let us see how to go about doing this. So, in place of v i I will plug in this expansion in terms of the basis. So, I have summation over i, eta i summation over j, alpha i j e j, right. So, i goes from 1 to n plus 1, but j goes only from 1 to n, right. It is an n-dimensional space. So, the number of vectors in the basis are going to be n.

Now, but this is an expansion of the 0 vectors in terms of the given basis, right. And every vector has a unique expansion therefore, the coefficient corresponding to every basis vector itself must be 0, right. So, the uniqueness of this expansion forces each of the coefficients separately to be 0. So, we have n homogeneous equations in n variables, right.

So, let us write that down. So, we have, you know I am collecting first of all I am collecting the coefficients corresponding to e 1 that will be alpha 11 eta 1 plus alpha 21 eta 2 all the way up to alpha n plus 1 comma 1 eta n plus 1 equal to 0 that is the first equation.

The second equation will collect all the you know the sum of the coefficients which correspond to the vector e 2. And that also has to be 0; third one so on all the way up to n plus 1. So, we have eta 1, eta 2, eta n plus 1. We have n plus 1 unknowns and only n equations and all of these are you know have 0 on the right hand side. So, this is a set of homogeneous equations.

So, we have seen that every homogeneous set of equations necessary is consistent. So, for sure it has at least the trivial solution. The trivial solution is where all the etas are 0, but that is not enough for us. If you want to find a non-trivial solution because if there is only a trivial solution that means, we are not able to show the linear dependence, right.

So, but the point is that since it is a homogeneous set, we can add one more equation for free, right. I mean it is a homogeneous set and we can add one more equation for free and keep it a homogeneous equation, but now we have n plus 1 equations in n plus 1 variables, right.

So, you see that now we can bring in our determinant rule, right. Whenever you have a homogeneous system of equations which is also square, right. So, now, we have converted this non-square system of homogeneous equations into a square system of homogeneous equations for which we had a very simple rule. If you want to find a non-trivial solution for this, then all you need to show is that the requirement is that the determinant of the coefficient matrix must be 0. Now, which is evidently true in this case, right.

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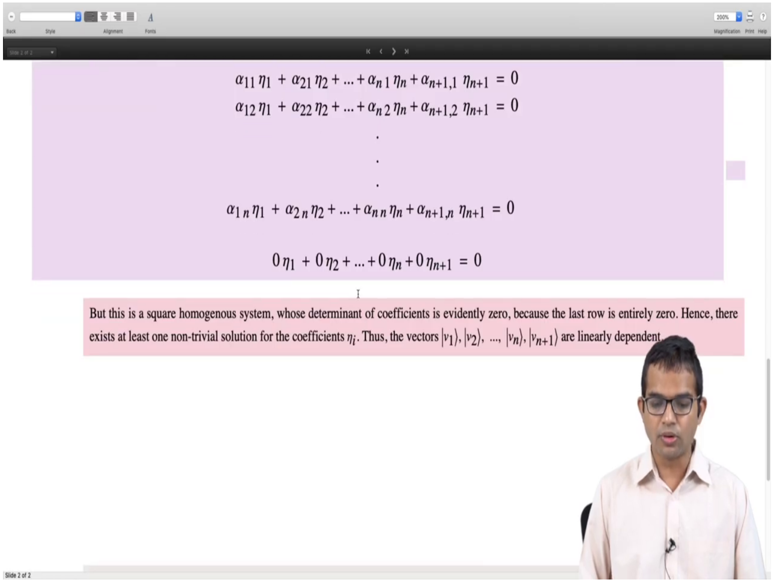

So, if you forgot this part, you should go back to the video from a few lectures ago and review it, right. If you have a square system of homogeneous equations and whose determinant of coefficients is 0, then for sure you have at least one non-trivial solution for the coefficient theta n.

Thus, the vectors v 1, v 2, all the way up to v n n plus 1 are actually linearly dependent, right. So, that is all, right. So, we have managed to show that any arbitrary set of n plus 1 vectors in an n-dimensional space must be linearly dependent, right.

So, in the light of this discussion you can go back and think about the statement I made in the previous lecture, where I said that the number of linearly independent rows and the number of linearly independent columns of a matrix must be the same, right. So, you might think that, I can come up with or construct a matrix with a very large number of columns, but just a much smaller number of rows, right.

Suppose we had 3 rows and 100 columns. So, you might ask how can you know the maximum number of linearly independent rows is just 3? So, even though I have a 100 columns I will be able to get only 3 linear maximum, at most 3 linearly independent columns, right. So, because the theorem of the equivalence of the row rank and the column rank will force this.

And so, in the light of this result that we have just proved you might want to go back and think about this, so the moment you have you know each of these columns although you have a large number of columns, they have only 3 elements in each column. So, therefore, you know the moment you have more than 3 there is going to be linear dependence, right.

So, there is a connection between this result and you know the result from the previous discussion and I would like you to think about this.

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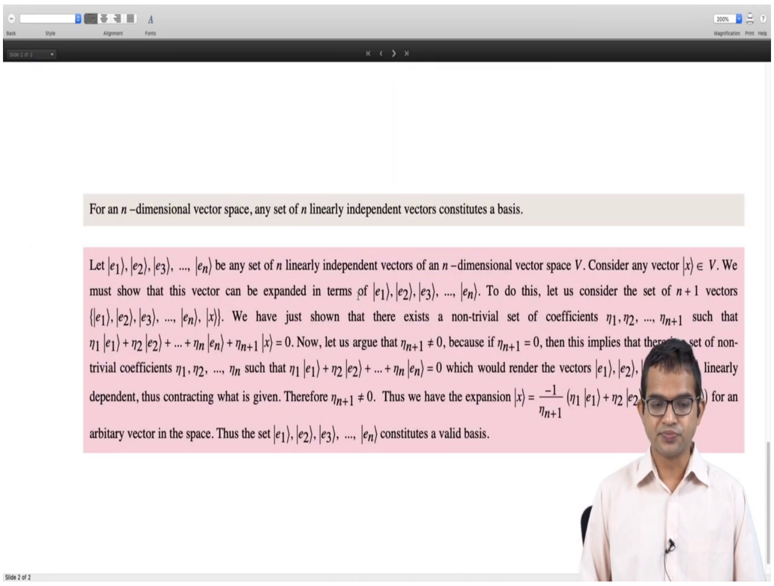

And before we end this lecture, there is one more result that I want to discuss. Suppose you have n-dimensional vector space then any set of n linearly independent vectors constitutes a basis, right. So, we saw for the two-dimensional space you could think of e x and e y, or e x and e x plus e y or e x plus e y and e x minus y. Any of these constituted a basis.

So, in fact, you can show like we are going to do now that any n linearly independent vectors for a n-dimensional space is a basis, right.

So, how do we see this? So, let e 1, e 2, e 3, so on up to e n be some arbitrary set of n, their key point is that they are all linearly independent vectors of an n-dimensional vector space. Now, what do we have to show? We have to show that we should be able to expand. So, this set of vectors is going to have to span the space, right, because they are already linearly independent if they manage to span the space, they are a basis.

Now, consider any vector x which is an element of v, we must show that this vector can be expanded in terms of this. So, let us consider you know the set of n plus 1 vectors, right you have e 1, e 2, e 3, all the way up to e n and let us also put x into this box and we have a set of n plus 1 vectors, and in an n-dimensional space.

But we have just managed to show in the previous result that any n plus 1 vectors have to be linearly dependent in an n-dimensional space, right. Now, that means, that we will be able to find these coefficients eta 1, eta 2, all the way up to eta n plus 1 such that you know non-trivial coefficients, such that sum of this object is going to be 0, eta 1 e 1 plus eta 2 e 2 so on up to eta n plus 1 x is equal to 0.

Now, all we have to do is show that eta n plus 1 is not equal to 0 and then we are done as you see in a moment. Now, why is eta n plus 1 not equal to 0? Because if eta n plus 1 equal to 0, then it means that eta 1 e 1 plus eta 2 e 2 all the way up to eta n e n is equal to 0, right.

And if, and also, eta 1 eta 2 eta n are non-trivial coefficients, if this is true then this will imply that the set e 1 e 2 all the way up to en itself is linearly dependent, but that is not true. We have given that e 1 to e n are a set of n linearly independent vectors. Therefore, eta n plus 1 is nonzero.

Now, why is this important for us? It is important because we would like to divide throughout by eta n plus 1, right. If we do this then we can express x as minus 1 over eta n plus 1 times you know this eta 1 e 1 plus eta 2 e 2 so on all the way up to eta n en . So, that is we have managed to show that any arbitrary vector in your vector space can be expanded as a linear combination of these n linearly independent vectors. Therefore, you know this set is a basis, right.

So, that is one more result which follows directly from the previous result that we described; earlier which in term followed from you know ideas from matrices. And you know all these results are cleverly and intimately linked together. So, if you are confused about some of these ideas, you should go back and review some of the ideas from the earlier lectures. That is all for this lecture.

Thank you.