**Mathematical Methods 1**
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**Linear Algebra**
**Lecture - 02**
**Linear vector spaces**

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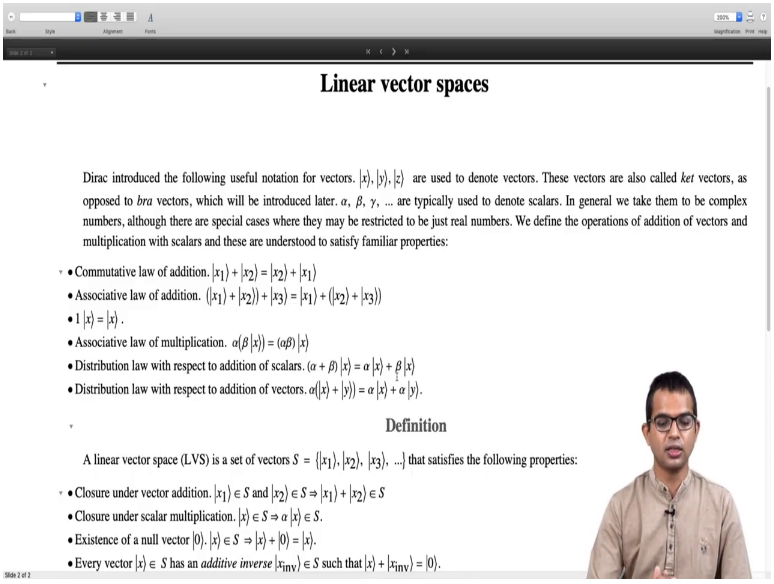


So, in this lecture, ok; so, it was Paul Dirac who came up with this very useful notation for vectors, right. So, we will also follow his approach and you write vectors which are enclosed inside this you know there is a straight line followed by a useful name like x, y, z, so on and then there are these angular brackets which you know which have this orientation, right. So, these will be used to denote vectors, right.

So, what goes inside is something which is an instructive aspect of the vector, right. It is not, there is no hard and fast rule what should appear inside here, but vectors will be enclosed inside these you know this symbol. Now, we will also use another kind of these angular brackets which are in the opposite direction which will be called bra vectors.

We will introduce these later on. So, Dirac used this in the context of bra and ket. So, he broke this word bracket and then we will see that these brackets also appear later on, and you probably have seen this if you have taken a course on quantum mechanics, right.

So, let us start with just these ket vectors, at some later point we will introduce the notion of a dual space and bra vectors will come in later on, ok. So, and typically we will use alphas and betas and gammas and so on to denote scalars, right. So, for our purposes scalars will be either complex numbers or in some cases they may be real numbers, right.

So, mathematicians would be more careful and or they would want to keep this even more general, and sort of think of these scalars as belonging to some field and you know there is a more abstract way of defining it. But as physicists it is sufficient for us to think of these scalars as you know either real numbers which are a special you know subset of the type of problems or vector spaces which we will be looking at. But in general alphas, betas, gammas will be complex numbers.

So, we define you know vectors as you know these objects which are denoted using these symbols and they satisfy certain you know rules; there are these operations: what can you do with the vectors you can add vectors, right. So, there is also the operation of addition that must be defined. If you are thinking of a vector you must be able to take any two vectors and add them, right. So, there is a notion of addition which must be specified, right.

And an addition must be such that it must be commutative, right. So, we are familiar with two-dimensional vectors, three-dimensional vectors, we already have this parallelogram law of addition, and so on. But in general addition can be some other kind of operation provided it is commutative in nature, right.

So, if you add x 1 and x 2 in one way in one direction or in the other direction, the result should be the same, you must get a vector which is the same whether it's x 1 plus x 2 or x 2 plus x 1, right. It must be associated, which is the statement that x 1 plus x 2 you know you put a bracket around them, you first perform this operation you get a vector and then you add this to a third vector x 3.

So, it does not matter whether you do it in this order or if you first add it x 2 plus x 3 and then you know you added x 1 after you have performed this operation x 2 plus x 3, right. So, notice that commutation and associative properties are slightly different, right. So, they are not they you know you cannot go from one to the other, they are independent properties, right.

So, there are spaces where commutativity may be relaxed, but associative law is very fundamental to many of these algebraic structures, but for vector spaces commutativity is also assured, right. So, we are dealing with addition. And also there are these other properties which connect to multiplication, multiplication with scalars, right.

One you know axiom if you wish is if you take any vector and multiply with the scalar 1, you are going to get back the same vector, it sounds obvious, right, but it needs to be encoded as a property, right. And many of the other properties of these vectors and vector spaces will follow if you start with some basic set of axioms and one of them is this obvious sounding property, 1 times x is x.

And then there is the associative law of multiplication. If you take two scalars and if you just first multiplied these two scalars and then multiply it you know this product with a vector it is the same as taking the vector and multiplying you know one of these scalars with the vector and then you multiply the resulting vector with another scalar.

So, there is associativity here with respect to multiplication of scalar. Then there are these two distributive laws which must also be satisfied, right. So, if you took two scalars, if you added them and then multiplied them with a vector you will get the same answer if you multiply it you know each of these scalars with the vector separately and add the resulting vectors, right. And that is the distributive law with respect to addition of scalars.

Then there is a distributive law with respect to addition of vectors. If you took a scalar alpha and multiply it with the sum of two vectors x and y, you are going to get the same answer if you first multiplied the scalar with these two vectors x times alpha times x and alpha times y, and if you added these two vectors you would get the same answer, right.

So, these are also laws which would appear obvious in some sense because we are familiar with you know 2D vectors and so on. But if you have these properties laid down, then we are ready to define formally what a linear vector space is. A linear vector space is simply a collection of vectors that satisfies these properties. So, the two key properties are closure under vector addition and closure under scalar multiplication, and then you need two more properties which I will explain.

So, closure under vector addition simply means that if you add two vectors which belong to the space you cannot get a vector which is outside of this, you will stay within the same space. That means, the sum of two vectors drawn from the space will yield a vector which also belongs in the same space. That is what closure means.

And likewise if you multiply with any scalar, right, the scalar could be a complex number or a real number for all you know all the examples that we will look at, they will turn out to be either the complex field or the real field. If you multiply it by an arbitrary complex number or an arbitrary real number any vector remains a vector which belongs in the same space, right if it is a linear vector space.

And it is important that a null vector is present in the space, right. A null vector is something which when added to any vector will leave that vector unchanged, right. So, x plus 0 will be x, right. So, and finally, every vector which belongs to the space will have an additive inverse, right. So, such that if you add this vector with its inverse you must get the null vector, right.

So, for those of you who have some exposure to groups you might recognize that in fact, these vector spaces will form groups under addition, right. So, if you have not seen groups and if the statement means nothing to you, not to worry; so, that is just a sort of a comment on the side.

But the point is that a vector space is a broader algebraic structure which also has not just you know this composition law which is called addition, but it is all it also has you know scalar multiplication and you have all these properties associated with addition and multiplication.

And so, we will see that with this sort of simple definition many consequences will follow. So, before we do that let us look at a few examples, right.

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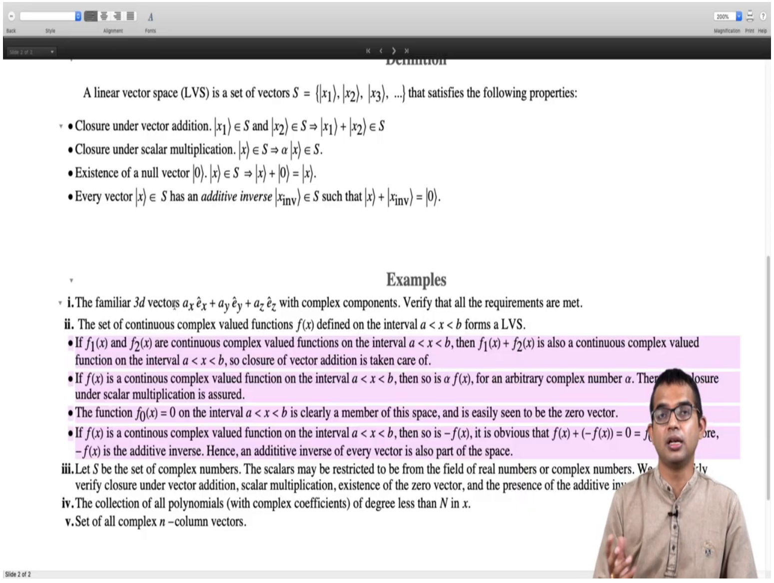


So, the familiar 3D vectors, right, 3D vectors or 2D vectors. Of course, if you take any 3D vector and add another 3D vector, addition of course is this parallelogram law of addition that we already know, you are going to get a vector which is also a 3D vector, right. Closure under vector addition is obvious, closure under scalar multiplication. So, if you have complex coefficients, if you multiply it by another complex number you are going to get another vector with complex components, right.

So, if you are using complex components perhaps the algebraic approach of adding component wise is a better approach, but if you had real numbers you can also visualize them, you can think of it as you know vectors in 3D space. And, then you have this complicated parallelogram law which is going to leave the vector, giving you a vector which lies in the same space. Addition is a vector addition is a closed operation.

Likewise, there is closure under scalar multiplication. So, the existence of a null vector is also obvious. So, the null vector is simply a vector where a x equal to a y equal to a z is equal to 0, right. So, that is the null vector. And every vector of course, has an additive inverse, right. So, if you take a vector made with the component a x, a y, a z its additive inverse is going to be minus a x minus a y minus a z those would be the components, right. If you add these two vectors you are going to get the null vector.

So, let us look at another example which is perhaps you know a set of quantities which you might not think of as vectors, but they also would classify under this definition, right. That is the whole point of making this notion more abstract is to be able to put in you know things into this box which you traditionally perhaps not think of as vectors, they are also vectors, right.

So, think of a set of continuous complex valued functions f of x, they are defined on some interval a to b, now they also form a vector space. So, every one of these functions is a vector. If you take any two complex continuous complex valued functions, if you add the two you are going to get another continuous complex valued function on the same interval. So, closure of vector addition is taken care of.

So, likewise if you multiplied some function f of x of this kind with some arbitrary complex number alpha, you are going to get another function which is also a continuous complex valued function on that same interval. Therefore, closure under scalar multiplication is also taken care of.

Then, you can think of this function f0 of x which is just 0 in the entire interval, right. And it is definitely a member of this space and it is going to serve the role of the 0 vector or the null vector. And finally, if you take any vector which belongs to the space that is some of this function f of x, you can also pull out another function which is minus f of x which is also going to be continuous complex valued and if you add these two you are going to get the null vector which is 0.

Therefore, every vector has an additive inverse. So, all the requirements are met and therefore, this is a linear vector space. So, here are a few more examples which I have put down. So, the set of complex numbers, you can think of every complex number itself as a vector, right. So, we may restrict the scalars to be you know real numbers or complex numbers also you can take them, right.

So, every complex number will have two components, right. And if you add any two complex numbers you are going to get another complex number which lies within the same space. So, vector addition is just you know the addition operation is the usual addition that we do of two complex numbers.

And closure under vector addition is met, scalar multiplication also met, existence 0 vector is simply the 0 complex number, and the additive inverse is for any complex number z minus z is going to be its additive inverse. So, just the set of complex numbers itself can be thought of as you know vectors in this abstract space.

Also, the collection of all polynomials of degree less than N in x, if you take a polynomial whose degree is N or less than N and add it to another polynomial whose degree is less than N or equal to N, you are going to get a third polynomial whose degree is also going to be less than or equal to N. So, closure under vector addition is met. So, likewise you can convince yourself that closure under scalar multiplication, existence of a null vector, and existence of additive inverse all these properties are met.

So, indeed collection of polynomials of degree less than N in x is also a linear vector space. Likewise, the set of all complex n-column vectors; so, think of just a vector with you know just n components. So, it is a little bit like you know the first example. The first example was just 3D vectors which you know connected to our experience, but you can generalize this to n-column vectors, so it will have n components and then you can quickly verify that all these properties of a linear vector space are satisfied. So, these are all some examples.

So, the point to note here is we started with familiar 2D vectors or 3D vectors, thinking of them as objects which have you know magnitude and direction and which add in a certain way and we have generalized it. And now we can think of even objects which you know traditionally you might not think of them as vectors, and they also satisfy these properties. And many times these other quantities will have other kinds of structure as well and still they also play the role of vectors and they form a vector space.

So, the whole point of you know this approach here is to abstract out these properties and just ask what all consequences come from looking at these abstract properties, right. So, that is what you know the philosophy here is. And we will look at many of these properties as we go along in the next many lectures on this topic.

Thank you.