Mathematical Methods 1
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Linear Algebra
Lecture - 17
Square System of Equations

So we saw the Cramer's rule which gives you a prescription for finding out the solution of a Square System of Equations right. So, when you have n equations in n variables linear equations Cramer's rule tells you how to find a unique solution if it is a unique solution.

So, we saw that the critical criterion there was that if the determinant of this matrix of coefficients was nonzero, then Cramer's rule holds. But we will look at this a little more closely and see what happens in particular when your determinant of these coefficients is not 0 right. So, let us analyze this a little more in this lecture.

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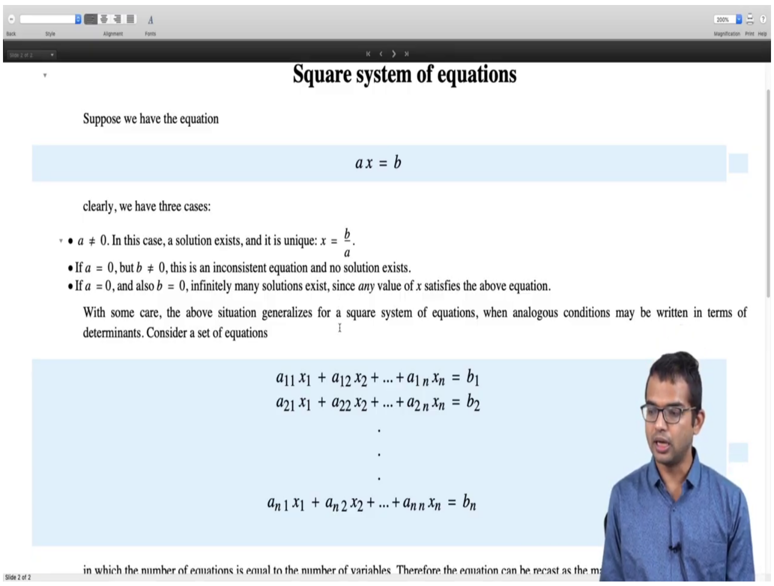

So, let's start with a very simple problem. Suppose you have given this equation a x equal to b right involving just numbers right, just I am not thinking of them as vectors or matrices or anything. So a x equal to b, so it is evident that in this system of equations you want to find all x such that a x equal to b.

If a is not equal to 0 then for sure there is a solution and it is a unique solution and that solution is simply given by x equal to b by a. You do not even worry about what b is in such a case, if a is not 0 the answer is it there is the solution exists and the solution is unique right it simply given by x equal to b over a.

But, if a equals 0 then it is a problem right you have to be careful right. So, you might think we are having a very pedantic discussion, why would we even worry about a solution for a system where a is equal to 0 right. So, it is useful for the purpose of generalization right. When we are looking at the system of equations we will see how these ideas can be generalized.

Now, if a is equal to 0 then we have 2 cases if a is equal to 0 and b not equal to 0 or a equal to 0 and b also equal to 0 if a is equal to 0 and b not equal to 0. That means, the left hand side no matter what value of x you choose the left hand side is going to go to 0 and you are trying to equate it to a value b which is nonzero.

So, that is an inconsistent equation you will never be able to find an x for which this will hold right. So, this is the case of an inconsistent equation and there is no solution. But on the other hand if a is 0 and b is also 0 then it does not matter what the value of x is, every value of x is a solution. So, you have infinitely many solutions. So, it is consistent and in some sense it is trivial because it has infinitely many solutions right.

So, we will see how these ideas will generalize when we go to a system of equations, when we have a square system of equations we can work with determinants right. When you have a non-square system then we will have to work with the rank of the matrix and work with row reduction and so on, which is in fact more general than this.

But really ultimately even a square system can be solved with the other method and they all give you the same answer. But the determinant method is a little more limited, in the sense that you are restricted to square systems and let us see how this generalizes.

So, consider a system of equations where it is a square system. So, you have x 1 x 2 all the way up to x n, these are the n variables and you have n linear equations. So, this is a very common situation where you have as many equations as there are unknowns; so, this is an important special case and let us see how the determinant approach is able to answer this question of what is the status of the set of equations.

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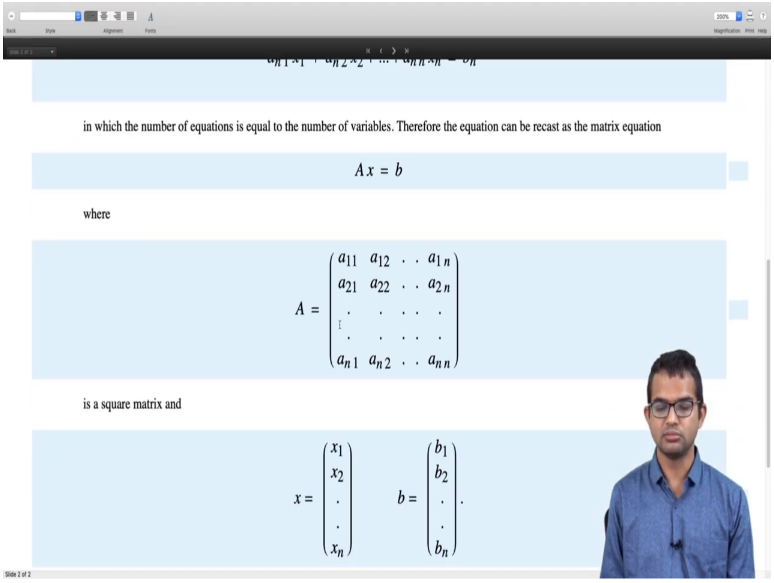

So, if you have the number of equations equal to the number of variables, that is the situation that we are considering. So, we can rewrite this equation as Ax is equal to b, where A is this matrix you know of coefficients then x is this vector which is also a matrix in some sense right; a vector is also a matrix but it is better to call it a vector here right.

In you know in line with the way we have been dealing with vectors in our you know abstract vector space context and we will get back to that discussion soon. So let us think of these x as vectors b is another vector - it is like the resulting vector you can think of a as a transformation right.

So, we are getting a little bit ahead of ourselves. So, that will be the topic that we will come to in a few lectures from now, which is you know how a transformation will take a vector and convert it to another vector. That is what is going on here really right.

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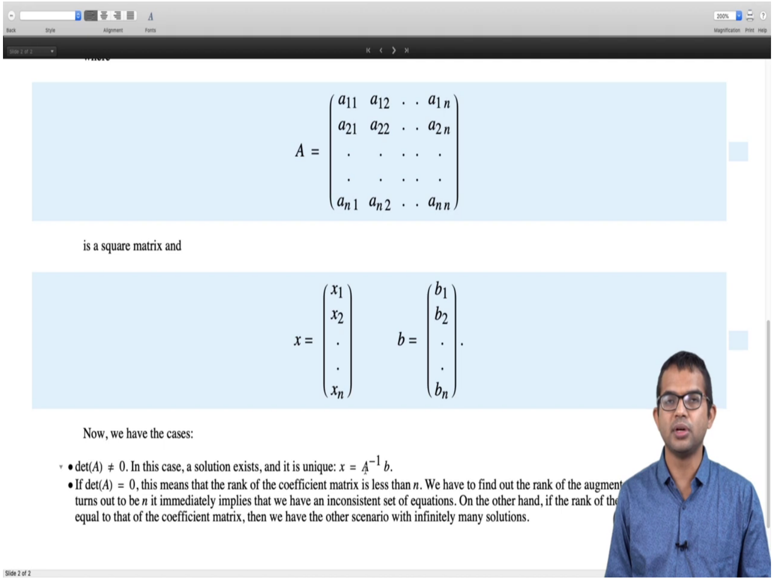

So, now we have the two cases here. Actually the second case has many possibilities, but I am just thinking of it as either determinant A is nonzero or determinant of A equal to 0. If the determinant of A is nonzero, then the inverse of this matrix will exist right. I have not explicitly defined what the inverse is, but you know what the inverse is right, from some courses in the past.

So, you know that the inverse of a matrix of a square matrix is a matrix which when multiplied with the original matrix will give you the identity matrix. Whether you multiply it in one way or the other it is a commutative thing A times A inverse or A inverse times A, if it can give you identity then such a matrix is inverse.

Now, in this case if you are able to find the inverse then x is equal to A inverse b you can just go ahead and write down. You multiply both sides with A inverse in this equation, then you see that the left hand side will become identity A times x right hand side will become A inverse b right.

So, this is very similar to this case we had when you had just numbers x equal to b A inverse b right, that is what you have here you have to be careful about the order from which you are multiplying right. When you go to matrices, basically the answer is unique and it can be written as A inverse b. But this is also equivalent to the answer you would get by invoking Cramer’s rule.

So, we also saw that Cramer’s rule holds for such a system and you can just write down the solution in terms of Cramer's rule. Whenever the determinant of A is nonzero, if the inverse exists you can either find the inverse and compute this a product of matrices. The matrix with this vector b or you can just invoke Cramer’s rule and find the determinant of in the numerator determinant at the denominator and divide.

But what happens if the determinant of A is 0. Now this means that the rank of the coefficient matrix is less than n right. So, we saw how you know the rank of a matrix can be thought of in terms of determinants right. So, the coefficient matrix is square in this case. So, if you have found that the determinant of this n by n matrix is 0, that is the only n by n matrix available because it is a square matrix. So, definitely the rank is less than n right.

So, now what about the rank of you have to then find the rank of the augmented matrix, which is this entire thing right without the variables. Now, for the augmented matrix if the augmented matrix is a rank you know full rank matrix, which means that it is if it is n then you are in trouble. That means that the rank of your augmented matrix will not match the rank of your matrix of coefficients, which means that there will be no solution; it is an inconsistent set of equations.

But on the other hand if you are able to show that the ranks of these two are the same, then you will be able to find in fact infinitely many solutions right. So, there will be some degrees of freedom - how many degrees of freedom are free will depend on what is that rank right.

So, not only you know first you have to verify that the rank of the augmented matrix is equal to the rank of the matrix of coefficients and then find what that value is. And depending upon what that rank is with respect to the total number of unknowns you will have so many free degrees of freedom right. So, this is also something that we have discussed in the past right.

So, this was a short lecture describing how one can think of a square system of matrices and how you know the fate of such a system can be determined from the determinant. And then once again the rank analysis may need to be done if the determinant of A is equal to 0.

So, in that sense the rank method, row reduction method is the superior one because it is the more general one right. If you are able to find the determinant and if it is nonzero, then you will immediately have access to the unique solution right. So, that is all for this lecture we will consider another special case in the next lecture.

Thank you.