Mathematical Methods 1
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Linear Algebra
Lecture - 15
The rank of a matrix using determinants

So, we have seen the notion of the rank of a matrix. So, a matrix in general does not have to be a square matrix. So, it can be a rectangular matrix. We have seen how you know the notion of a determinant is applicable to a square matrix. So, in this lecture, we will see how it is possible to work out the rank of a matrix and arbitrary matrix, square matrix of rectangular matrix does not matter using determinants right.

So, we have already seen how one can get the rank of a matrix using row reduction, but it is also possible to you know evaluate this in terms of using determinants of certain sub matrices right, so that is the subject of this lecture.

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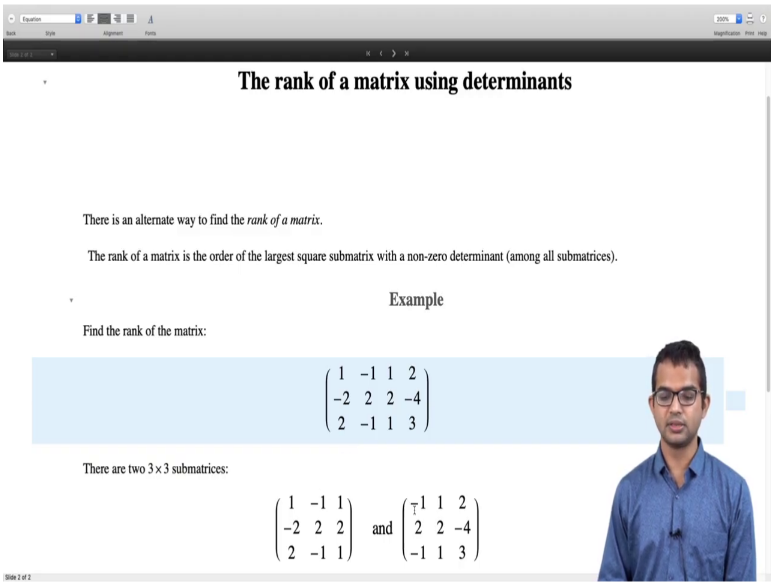

So, the rank of a matrix is simply given by the order of the largest square sub matrix with a nonzero determinant right. So, it's best explained with the help of an example right. So, you consider some matrix. I have a matrix with three rows and four columns here. And I want to find out what its rank is. Of course I can do row reduction - you know the drill how to go about finding the rank of this matrix.

Now, there is another way to find this, which is to study all sub matrices. How do you find sub matrices right? So, sub matrices are you know a sub matrix is obtained from a given matrix by deleting some rows or some columns. So, in this case, for example, if I were to delete the first column, I am left with this sub matrix. So, in fact the first column itself is a sub matrix right, but we are interested in square sub matrices.

So, we would be interested in this matrix, for example, there are only three only two 3 by 3 sub matrices right. So, if you are given a 3 by 4 matrix right, then the maximum rank that this can have is 3, which is the minimum of the two dimensions of a matrix right.

This is something that you can convince yourself if you think about it for a moment, but it is also evident from here right. So, we have said that the rank of a matrix is the order of the largest square sub matrix with a nonzero determinant. So, there are only three or only two 3 by 3 sub matrices here.

So, let me highlight these two 3 by 3 matrices. So, if I am able to find either of these to have a determinant which is nonzero, then I can immediately conclude that the original matrix has ranked 3 right. But on the other hand, if I find that both of these determinants are 0, then I will look for 2 by 2 matrices. Is there even a single 2 by 2 matrix whose determinant is nonzero? If that exists, then the overall matrix will then have the rank 2 and so on.

So, let us do this. So, you may want to pause your video and ask yourself, “What are the determinants of these two matrices?” Can you work this out without doing any elaborate calculation right?

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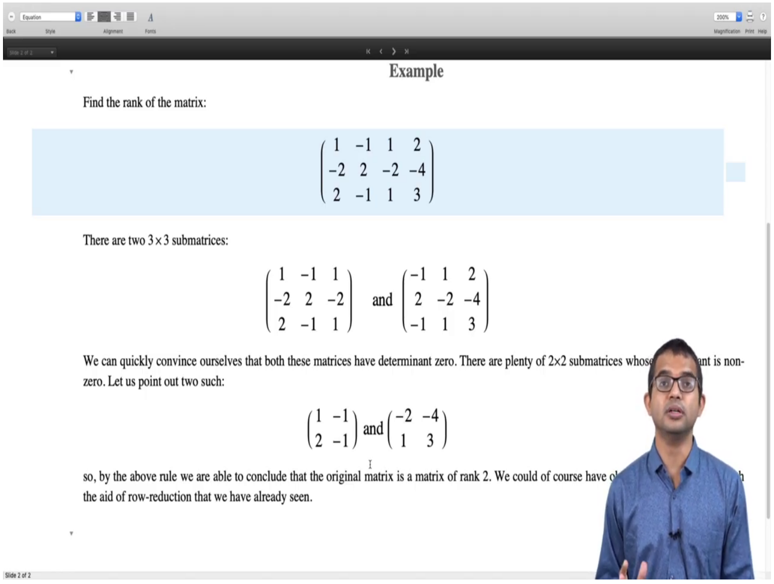

So, we can quickly convince ourselves in fact that both of these matrices have determinant 0 right. Why is that? So, if I look at the first row of this first matrix and the second row, if I take this and multiply by minus 2 ok. So, I wanted to arrange this in such a way that the second row is a factor of this. If I do 1 into minus 2, it is this. So, I want to change something here. So, let me redo this. So, I will recast this matrix as minus 2 here and minus 4.

So, I want to do this because I want to arrange it in such a way that the second row is a factor of the first row. So, that just to illustrate how this does not have a rank 3 right. So, let me also be careful about 2, minus 2, and minus 4. So, if I do minus 1 times minus 2, I get this; minus 2, I get this; minus 2, I get this right. So, there was a minus sign which was missing right. So, I had thought that I had arranged these two rows in such a way that the second row is a multiple of the first row right.

So, given that this is the matrix that I am interested in, then indeed you can quickly see that both these determinants are 0. So, then I go one order lower. So, I will look at all sub matrices, square sub matrices of size 2 by 2 all right, but I do not actually have to look at all of these. If I can find even one such 2 by 2 matrix whose determinant is nonzero, then I am done because this matrix is then going to have rank 2 and that is the case here right.

So, let me find for you - I have given here 2 ok. So, there is minus 2. So, it is minus 2, so minus 6. So, indeed both these so how did I get this first? Sub matrix so here I have you know pulled out these sub matrices just to show you what can classify as a sub matrix right. I have deleted the second row entirely, then I have deleted the third column and the fourth column. So, I have left with 1 minus 1, 2 minus 1, this is an acceptable sub matrix.

And likewise in the you know second example I have constructed this by deleting the first column second column and the first row. So, I have minus 2 minus 4 1 3. Both of these are 2 by 2 sub matrices whose determinant is nonzero, but actually you just need to find one such a sub matrix whose determinant is nonzero, and then you are done right.

So, it tells you that this overall matrix has the rank 2, but I mean just to eliminate the possibility of a 3 by 3 matrix. So, there you have to look at all possibilities right. If you are able to show that all the possible 3 by 3 matrices have determinant 0, then for sure the rank of your matrix is going to be less than 3.

And then if you want to establish that it is not even a rank 2 matrix or lower than like 2, then again you have to go over all the possible 2 by 2 sub matrices. But if you are able to find even one such sub matrix to be nonzero, then you can immediately conclude that it is a matrix of rank 2 right. So, this is a short lecture describing this technique of extracting the rank of a matrix using determinants.

Thank you.