Mathematical Methods 1
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Linear Algebra
Lecture - 12
Rank of a matrix

So, in the last lecture, we saw how there is a systematic method which allows us to bring this matrix into the so-called row reduced or row echelon form right. So, we will see you know the point of doing this is not really about solving equations as such which we all know how to do I guess from high school, but you know to extract an important quantity related to any matrix which is called the rank of a matrix right. So, we will define what the rank of a matrix is here and again look at some more examples of how row reduction can give us this.

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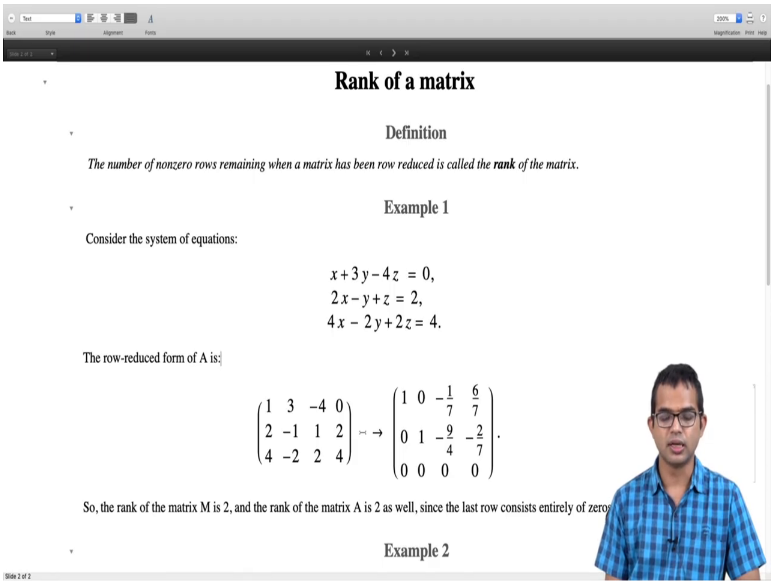

So, the Rank of a matrix is simply the number of nonzero rows remaining when a matrix has been brought into the row reduced form – the row echelon form right. So, you know in the prescription inherently all the rows which are entirely which consists entirely of zeros get pushed to the bottom. So, the top few rows which remain which are not entirely zeros right are the number of such rows is the rank of a matrix right.

So, let us look at an example. So, we have you know this system of equations x plus 3 y minus 4 z equals 0, 2 x minus y plus z equal to 2, 4 x minus 2 y plus 2 z equal to 4 right. So, how do we; how do we go about doing this row reduction? Right. So, I gave you the algorithm last time. So, let us quickly recall what we would do in this case right. I am not; I am not showing you all the steps involved right.

We start by writing this augmented matrix so called augmented matrix. So, I have 1 3 minus 4 0 2 minus 1 1 2 4 minus 2 2 4. So, you will have to fill in the details here right. So, the natural thing to do is to start since you already have some so-called pivot right, so the first row is already 1 and you just leave it as it is. Sometimes it is useful to exchange rows, that is also an allowed operation you can exchange rows, but in this case it seems convenient to just leave it as it is.

And then you start by looking at the second row you have 2, so you take the second row and subtract 2 times the first row, so then clearly you will get the coefficient 0 for the first term. And likewise you can do the third row minus 4 times the first row, so that is going to give you again a 0 coefficient here.

And now the next step will be to remove you know this coefficient which is there in the second and the leading coefficient in the second row will appear in the second term, so that you will have to you know use that to make the coefficient of the second term in the third row second column that also must be put to 0. If you do this carefully and then also demand that the second row, second column coefficient is made to be equal to 1 right.

So, then if you do this carefully you will find that in fact, there are only two rows right it is not going to have this you are going to be nothing in the third row. This is the row echelon form right. So, it is a row echelon form because the leading term in each row is just 1 or unless it is all zeros which is happening in the third row right.

And also if you have a leading you know this 1 appears in a certain column everybody else in that column must be 0 which is true, and that is it basically right. So, you have the row echelon form here. So, you should check that this indeed has been correctly obtained.

So, the point is that if I look at this over all augmented matrix A, its rank is 2 because there are only 2 rows which are not entirely known entirely 0. And likewise in fact the rank of the matrix M the matrix of coefficients is also 2, because to get you know to get the rank of this matrix you must consider only these elements right. And then you can quickly see that again this third row is just 0 as far as this matrix is concerned. So, indeed the rank of the matrix m is also 2 in this case.

And in fact, with a little more work or if you just observe carefully, you can read off the solution for this system of equations from the row reduced form you know by a method of back substitution. But in this case it will turn out that you cannot pin down the exact value for x, y and z, but you can write x plus x in terms of z, and y also in terms of z and that z is you know can take any value right.

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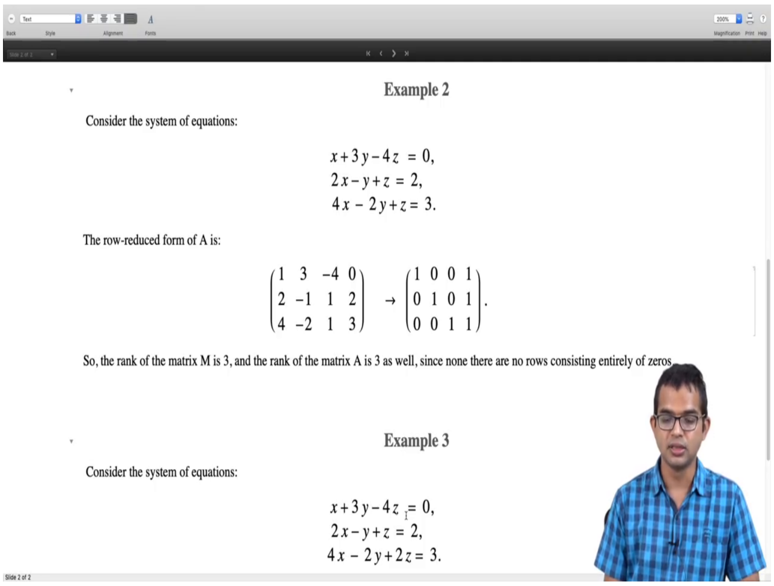

So, let us look at another example where I have basically doing the same set of equations slightly. So, instead of taking 4 x minus 2 y plus 2 z is equal to 4 the third equation first, and second equation have left them and changed. But the third equation I have made it 4 x minus 2 y plus z equals 3.

If I do this, now I see that the row reduced form becomes like this right. So, this is a little bit of work, but I am skipping the steps and I will allow you to play with this and work this out right. So, you have seen something like this since high school, so it should not be difficult to do it.

But what is important here is the method right. So, you must start with the first row, use the first row to make the second row the first coefficient go to 0, then again the third row make the first coefficient go to 0. And, then compare between the second row and the third row and it makes the third row second column coefficient go to 0, and then you know make all the leading terms 1.

And once you have done that, you subtract the corresponding coefficient from the other rows, so that wherever there is a leading element 1 in such a column every other coefficient must also be put to 0.

And once you have done this, you can recast it in the row echelon form which in this case turns out to be this very nice simple answer which is you know basically x equal 1, y equal to 1, z equal 1 which you can directly verify by checking. Once you know the answer, you can of course verify that it holds in all the 3 right.

So, let us look at the rank of matrix M and the rank of matrix A in this case. So, in this case, it turns out both A and M have the same rank, but the rank is now 3 right. There is no row at the bottom containing only zeros right. All the 3 rows which are there in this matrix are important, and they carry information that is what it means really right.

You may have lots of rows, but the amount of information contained in them may be equal to just a smaller number of rules. That is what the rank in some sense measures whether there is any redundancy in this matrix. And so the key point to observe here is that both the augmented matrix and the matrix of coefficients have the same rank, and that rank is equal to the number of equations that you have which is 3 right in this case.

Now, let us look at a third example where again I have treated the same set of equations the third equation I have made it slightly different right. So, I have x plus 3 y minus 4 is equal to 0 as it is 2 x minus y plus z equal 2 as it is. For an unchanging 4 x minus 2 y plus I am making it plus 2 z equal to 3 right it is different from example 1, example 2.

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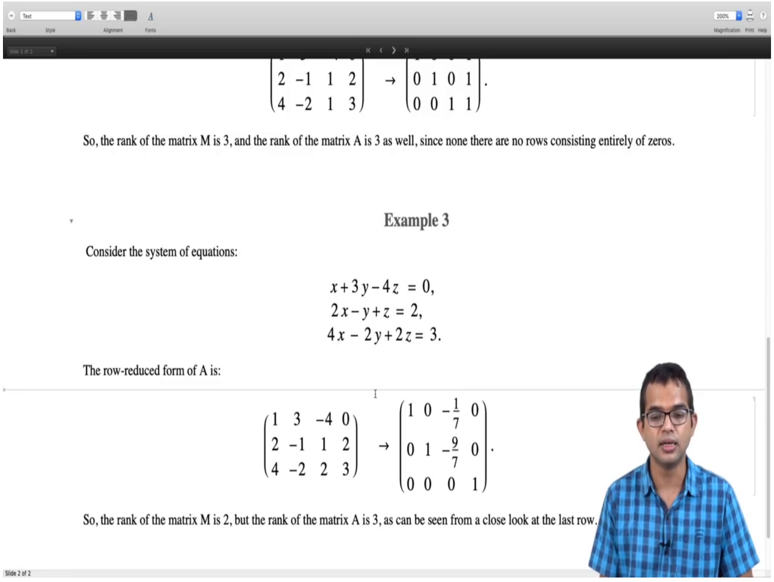

And in this case again you can start with the augmented matrix and go ahead and do the whole drill right, and then you will see. And so this you need to do it yourself right check that the row echelon form in this case gives you this answer which is different from both the forms that have been seen before in example 1 and example 2.

Here what you observe is in fact, if you look at let us look at the rank of these two matrices. So, the rank of matrix M is only 2 because if you look at these elements here you know this is a matrix M, this corresponds to matrix M.

You see that the third row contains no information; it is all zeros. So, it has only two rows worth of information and so it is a rank 2 matrix. Whereas, the full augmented matrix is a rank 3 matrix right because there is one sitting here. But if you look at it carefully what this means is we are saying is 0 equal to 1, and it is nonsensical right that is what the third equation is saying 0 x plus 0 y plus 0 z is equal to 1 which can never happen right.

So, in some sense you might guess that you know this inequivalence between the ranks of the 2 matrices is a signature of whether this set of equations has a meaningful solution of not right and which is indeed true. So, formally, we will state this result in the next lecture. That is all for this lecture.

Thank you.