Mathematical Methods 1

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Linear Algebra
Lecture - 11
Row reduction of matrices

So we started our discussion of linear dependence and linear independence of vectors. But before we continue along these lines, it is useful to take a slight detour, and discuss systems of linear equations which we are all sort of familiar with. So, the following few lectures will be in the nature of recall, but perhaps there are also some ways of thinking which are which you have not seen exactly in the way in which we will present it right.

So, and in the process we will you know describe something called the row reduction of matrices right. So, which is really an algorithmic way of solving for systems of linear equations.

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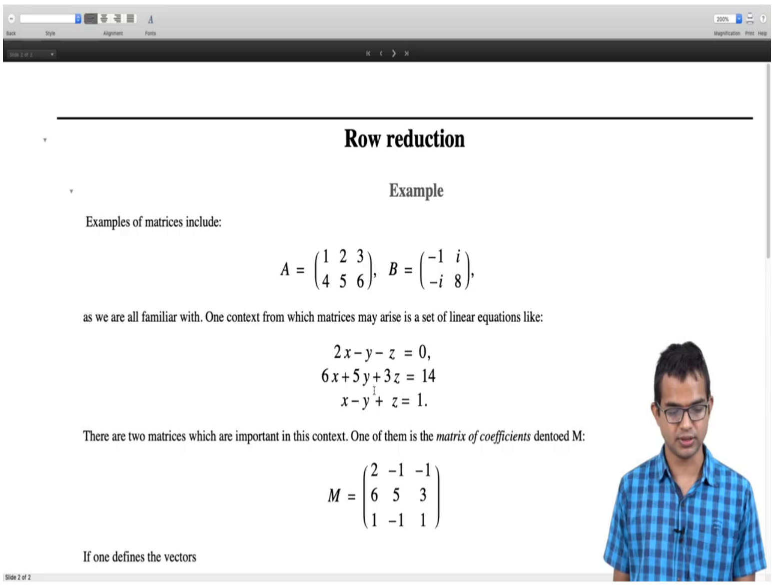

Let us begin our discussion of row reduction in this lecture ok. We are all familiar with the notion of a matrix right. So, examples of matrices include like what I have given here A and

B. You know matrices are just collections of numbers, usually which are there is a notion of a certain number of rows, there is a notion of a certain number of columns.

So, the i th column and the j th row is going to have an element, which is typically a number. It could be real numbers or complex numbers typically the kind of matrices that we are interested in right. And one context from which matrices come about maybe, if you have a system of linear equations like this I have chosen one particular example 2 x minus y minus z equal to 0 6 x plus 5 z plus 3 z equal to 14 and x minus y plus z equal to 1, right.

So, one matrix of immediate interest when you have given a system of equations like this is the so-called matrix of coefficients denoted M right. So, you just collect all these coefficients. I have a 2 I have a minus 1 and I have a minus 1 I put them in the first row 6 y 5 and 3 constitute the second row and 1 minus 1 and 1 forms the third row.

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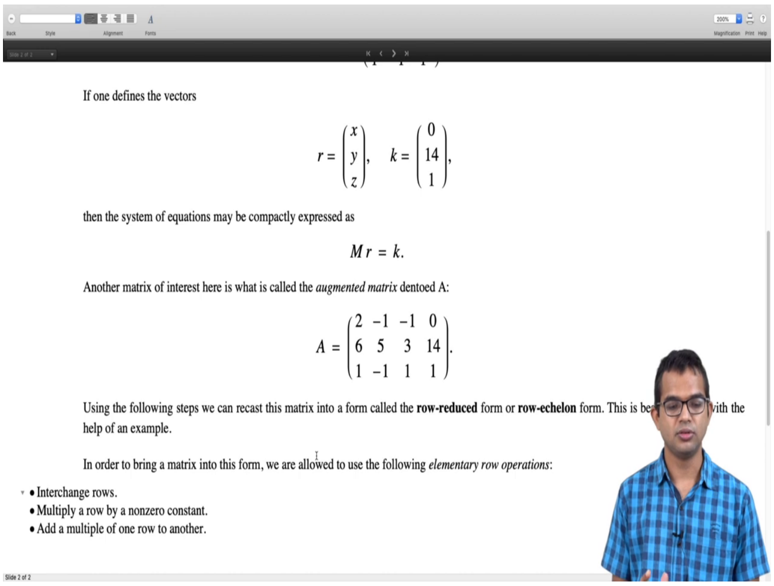

So, this is the matrix of coefficients for this system of equations. Now, it allows us to rewrite this system of equations in a compact way right. So, if we define these vectors r right x y z are you know are variables are unknowns for which, we want to come up with a systematic way to solve these and that can be written as a vector like this. And so, there is k is another vector which has these numbers 0 14 and 1 which appear on the right hand side.

We can be bundled together into another vector. If we do this we can go ahead and rewrite this whole system of equations as a matrix equation right. So, we are all familiar with how we can take a matrix and multiply it with a vector. And in general we know how to multiply a matrix with another matrix right provided you know they are compatible, right.

So, you are all familiar with this. So, we will not spend too much time going into the details of this right. So, we know how you can take the first row and multiply with this column right. If you know the number of rows of the matrix which appears to the right must be equal to the number of columns of the matrix, which are on the left if matrix multiplication must be well defined.

So, in this case of course, you know r is a vector right a vector is also a kind of matrix. In which the number of rows you know there are three rows in this vector r and which matches with the number of columns in M. So, its M times r is completely well defined. It will give you another matrix which in this case is just a vector right. So, matrix times the vector will give you another vector which will be a three row single column vector which is k right.

So, this is a compact way of writing down the same information that is already written given in this system of equations right. So, there is another matrix of interest which is called the augmented matrix. So, the augmented matrix is denoted as A and so, where you in fact also tag along this vector k also is appended into the matrix M itself and then you get the augmented matrix.

So, you have the M matrix, the first three rows and three columns you know that is where the matrix M is embedded. And then there is an extra column, which is also included to form the augmented matrix.

So, now, we will describe a method which allows us to recast a given matrix in a very convenient form right. So, you might be interested in understanding, you know how much information is contained in you know all these different numbers of equations given to you.

But, do they all you know contribute some extra bit of information or is there some redundancy or is there some inconsistency? You know these are the kinds of questions, which can be answered if we use this algorithm which will allow us to recast this matrix in the so-called row reduced or row echelon form, right.

So, it is best explained with the help of an example right. In order to do this let us just state that these elementary row operations are the operations which are allowed right. So, elementary row operations are operations which are invertible in the sense that by doing this there is no loss of information.

Whatever is there you know you have a set of equations. And then you are allowed to do these row operations and then come up with another set of equations really. And, there is no you know there is nothing essentially different between the first set of equations and the second set of equations that come about, when you do perform these elementary row operations because they are reversible right.

One is you can interchange rows right. So, it does not matter whether you had you know equation 1 appeared first and equation 2 appeared second or if you were to swap this equation. So, the content is the same right so, that is allowed. Second is you can multiply a row by a non-zero constant right that also seems clear that there should be no loss of information, you could always divide by that non-zero number again and get back the old equation.

And the third thing you are allowed to do is you can add you can take a multiple of one row and add to another row, it is going to be consistent right. So, you can take one equation multiply it by some constant factor and add it to another equation right or add it to a multiple of another equation. There is no loss of information right.

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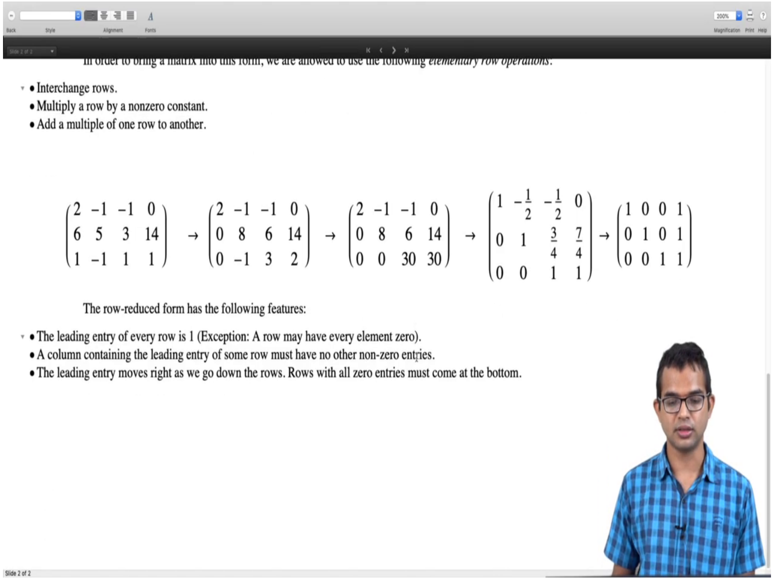

So, these three operations are called elementary row operations. With the aid of these elementary row operations, you can bring it into a form which is called the row echelon form. So, let me show you this example. So I start with 2 minus 1 minus so, this matrix that I have here 2 minus 1 minus 1 0 6 5 3 14 1 minus 1 1 1 right. So, we wanted to bring it into a form, where the leading entry of every row is 1 right.

So, that is one requirement, then a column containing the leading entry of some row must have no other non-zero entries and the leading entry keeps on moving to the right. So, let us just let me try and illustrate this.

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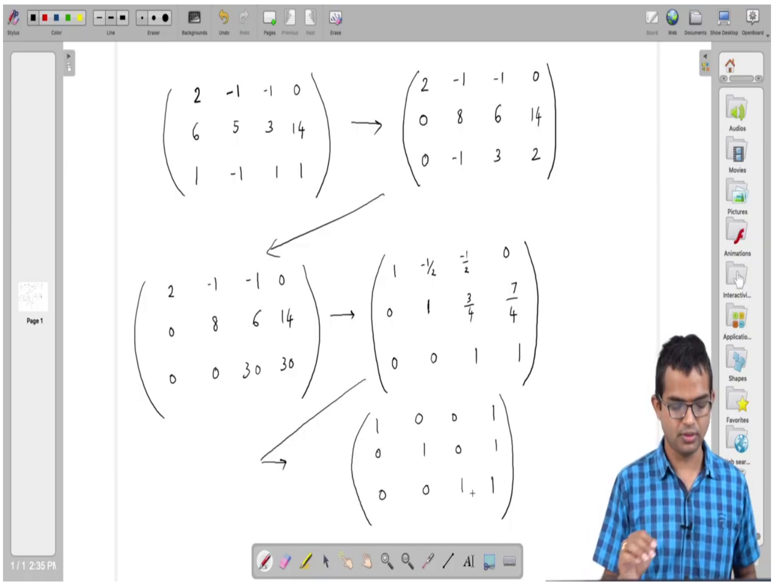

So, I have this matrix 2 minus 1 I have 2 minus 1 minus 1 0. Then, I have 6 5 3 14; 6 5 3 14, then I have 1 minus 1 1 1 1 minus 1 1 1 right. So, let me illustrate one step here explicitly and then I can explain what I already have there right. So, the idea is that I keep the first row as it is and then I will subtract 3 times the first row from the second row and replace the second row. So, the first row remains as it is 2 minus 1 minus 1 0.

But, then I take 3 times this 6 minus 3 times this so, the idea is that I want a 0 here so, 0 then so I have to do 5 minus 3 times minus 1. So, this becomes 8 and then I have to do 3 minus minus 3 so, this becomes a 6 and then 14 remains a 14. Because 3 times 0 is not going to change anything, then likewise I want to take this and convert this into 0. So, I will do 2 times this minus 2. So, I have 2 times 2 into 1 minus 2 is 0 2 into minus 2 minus 1 minus 2 minus minus 1 is minus 1 2 into 1 so, this becomes 3.

And then I have 2 into 1 minus 0 is just 2 right. So, that is all I have done. So, in the first step I managed to bring these two coefficients to 0, in the next step I will want to bring this also to 0 right. So, I can just add you know 8 times this plus this that will give me 0 already. So, let me do that also. So, from here I go here so, the first two rows remain unchanged, I have 2 minus 1 minus 1 0 0 8 6 14 0. So, then I am going to convert this coefficient also to 0.

So, I will take 8 times this whole row and add it to the second row. So, this goes to 0 8 into 3 is 24 plus 6 is 30 8 into 2 16 plus 4 is 30. So, this is also done, then I want to go to the next step which is I will just divide the third row by 30 right. Because, I do not lose anything with a multiplying throughout by a constant factor, in this case it is going to be 1 over 30.

So, this is 1 and 1 and here I am going to divide throughout by 8. So, I have 0 8 will become 1. So, the idea is that the leading coefficient along any row must be made equal to 1. So, this is 1 6 by 8 is 3 by 4 and 7 by 4. And the first row is going to become 1 minus a half minus a half and 0 and then finally, I want to you know in any column containing the leading 1.

So, like here I have this leading 1 and I do not want any other elements in that column. So, I have to take row 1 and then use the information in row 2 to make this coefficient all to also to go to 0. So, the way to do that will be to add half of the second row to the first row. So, if I do that then I have 1 remaining as it is, so, this will go to 0 so, then I have to add 3 by 8 minus 1 by 2 3 by 8 minus 1 by 2.

So, then I will have to repeat this operation at a later time. So, the point is that I will do this once and then I will have to do it again. So, I am going to skip this step and then just write down the final answer which you can verify. So, it is just going to be 1 0 0 1 and then you will get 0 1 0 1, I will allow you to do this operation right.

So, this is not quite the row reduced echelon form yet, you will have to do one more step and keep doing it carefully, but I have given the answer here in the slide form, which you can. I will allow you to complete the details. So, it is just going to be 0 1 0 1 0 0 1 0 1. So, this is actually the final answer. But, the row reduce form which is also the row reduced form and the solution both in this case, right.

So, there is this back substitution which is also being used in here right. So, first of all we can read out the answer from here right. So, the point here is not really so much to read out the answer as the method right. If you can bring it into this form where what are the requirements, one is that the leading entry of every row is 1 right. So, that is a check we have already got.

A column which contains the leading entry of some row must have no other nonzero entries right. So, that is where you have to do this extra hard work towards the end, which is to eliminate you know to bring these coefficients. So, when you go from this step to this step you have to make sure that this is 0 and also this is 0 and also this is 0.

In order to do this you have to add row 1 and row 2 in a suitable way. So, this part is not going to do anything to this coefficient 1 will remain as it is, you have to just carefully add these guys add row and row 1 and row 2. And likewise you have to add row 2 and row 3 in a suitable way such that this coefficient also made equal to 0. And then, if you do this you are able to get all these coefficients right. So, this is the so-called row reduced or row echelon form right.

So, we will see some consequences of this in the next lecture.

Thank you.