Mathematical Methods 1
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Linear Algebra
Lecture - 10
Linear dependence and independence of vectors

So, in this lecture we are going to look at the notion of dependence and independence of a set of vectors. So, if you are given a set of vectors you know what are the constraints that this set of vectors come up with, right. So, and this will be the concept that we introduce here will have very important consequences which we will explore in the lectures which will come ahead.

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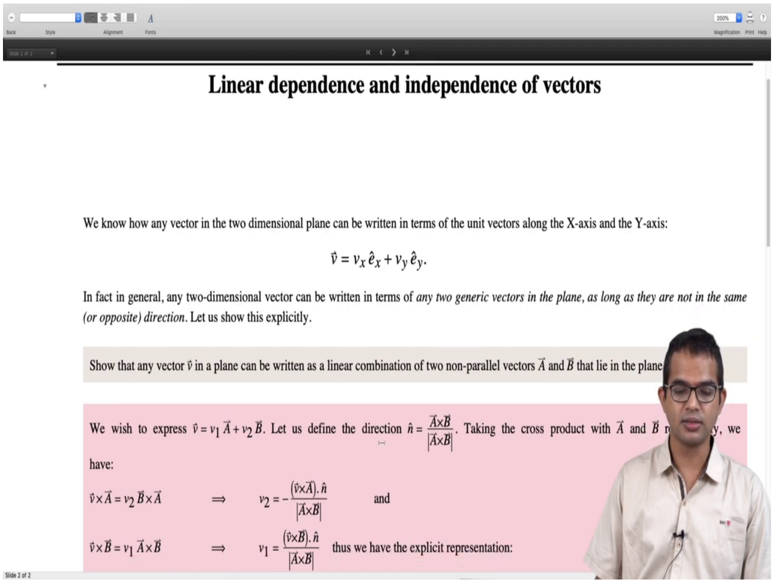

So, let us harp back to 2D vectors, simple two-dimensional vectors that we are all familiar with. So, if you have some any arbitrary vector in the two-dimensional plane, right which it is something that we take for granted, can be written as, you know can be expressed in terms of some scalar along you know some magnitude along the x direction plus some other scalar quantity times the y direction. There is a unit vector along the y direction and a unit vector along the x direction.

So, but in fact, any generic you know two vectors which live in a certain plane, right can act as you know these vectors in terms of which any other vector in that space can be represented, right. So, as long as they are not parallel, they are not you know in the same direction or in the opposite direction, it is possible to write down any other vector in that plane in terms of these two vectors, right.

So, in fact, let us go ahead and show this. I already have this statement that any vector in a plane can be written as a linear combination of two non-parallel vectors a and b, right. So, if you want to pause the video and try to work this out for yourself, covering you know whatever little it is already showing up in my solution of this, please do so, right.

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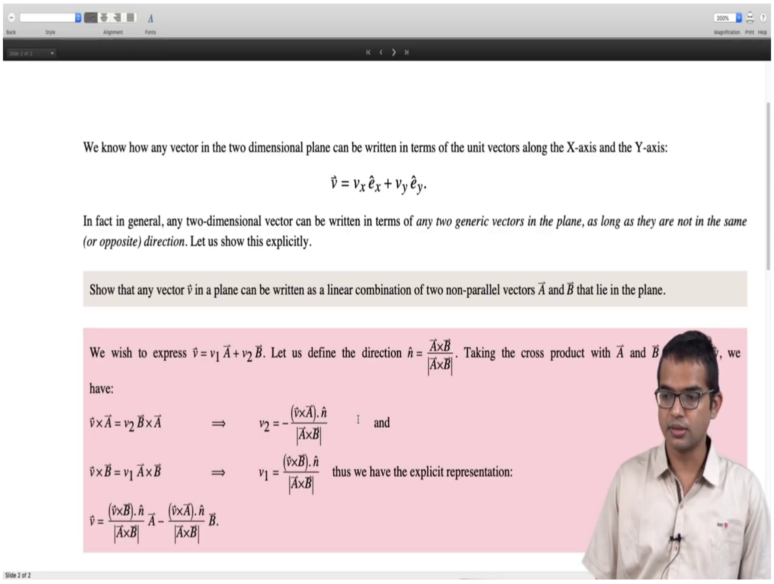

So, we want to write v as some v 1 times A plus v 2 times B, right. We do not know what v 1 and v 2 is, and we are only given some two vectors A and B arbitrary vectors, we are told that they are non-parallel, right. So, they are non-parallel. So, there is a cross product of these two vectors, right. So, I assume that all of us are familiar with the notion of a cross product.

I did not explicitly define it for two-dimensional vectors, although I did use the dot product because I think we are all sort of familiar with this from high school. So, let us take this for granted and say that given that these two vectors A and B are not parallel therefore, there is a cross product that we can find that is not going to be 0, right. And it is going to in fact, it is going to represent the direction perpendicular to the plane, right.

So, this is also something which I think is not surprising. None of what I am saying right now is going to be surprising, but I am just putting it in a form which is kind of abstract and which will open up you know ways for us to make more general statements for linear vector spaces, ok.

So, A cross B is a vector which is going to be perpendicular to both A and B, right and that is the point. So, if you are able to find a vector which is perpendicular to both A and B you have managed to define the plane. So, if there are two vectors which live in this plane and which are not parallel are enough to define the plane therefore, any other vector is going to be representable in terms of these two vectors, right. So, let us explicitly find that.

So, you take the cross product and divide by its magnitude. You only care about this direction, you have found this direction which is perpendicular to the plane. And now we can go ahead and use this information, right.

So, v 1; so, if I take the cross product of this vector v with respect to A, then only one of these two terms will survive. It is going to be v 2 times B cross A because A cross A is 0. And then we can take the inner product, the dot product, the dot product of you know this equation on both sides with this unit vector n.

And then of course, the right hand side is going to be a dot product of B cross A with respect to n which will just be equal to modulus of A cross B with a negative sign, right. So, A cross A, A cross B is equal to minus B cross A. And then you just bring that to the other side and therefore, you are able to write down this explicit expression v 2 is equal to minus dot product of this vector v cross A dotted with n and then you have to divide by A cross B.

So, given any vector v and I am telling you what its component along B must be. I have to take A cross product of this vector with A and then dot it with this direction which I have defined for you because I know A and B, and then I have to divide by this modulus of this A cross B. And likewise we can go ahead and compute v 1, the component of this vector v along the direction A, right. So, that is just given by the dot product of v cross B with n divided by the modulus of A cross B, right.

So, thus we have this explicit representation where I can write v as you know this component along A plus this component along B, right. It turns out to be negative and this is the way and I have written it. But basically the point is I have shown you explicitly what v 1 and v 2 are, right.

So, what it tells us is if you are looking at two-dimensional vectors any two non-parallel vectors are enough to give you every other vector in this space, right, although there are actually infinitely many vectors in this space, right. So, you are thinking of a box or a set of vectors, but the number of elements in the set is actually infinite.

But you do not really need an infinite number of elements to represent any of these, there is a, you just you can represent it in terms of just two vectors and those two can be any two vectors as long as they are not parallel and that is what this is telling us.

Of course, the coefficients that you can choose v 1 and v 2 are you know there are infinitely many possibilities. So, that is how you are able to generate an infinite number of vectors, an infinite space using just you know two vectors which form what is called a basis, right. I am getting ahead of myself, but we will define this a little more precisely in a future lecture.

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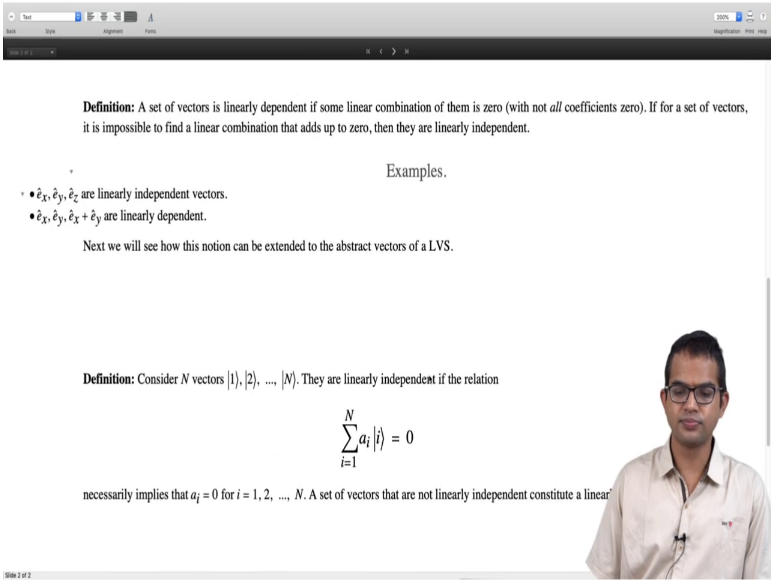

So, next I want to define the concept of linear dependence, right. So, I said you know I do not want these vectors which are parallel and so on, right. So, that is where you know the precise notion that we want to use here is that of linear dependence. So, a set of vectors is said to be linearly dependent if some linear combination of them is 0, right. If they are parallel then a linear combination of them is going to be 0, right.

If on the other hand, ok; there we go. If on the other hand if for a set of vectors it is impossible to find such a linear combination that adds up to 0 then they are linearly independent. So, for example e x, e y, and e z are linearly independent vectors, right. So, it is impossible to find a set of non-zero coefficients. You can of course, take 0 times e x plus 0 times e y plus 0 times e z equal to 0 that is possible.

But that the point is that whenever if at all there are if you are looking for some 3 coefficients a x, and a y, and a z such that a x e x plus a y e y plus a z e z if that has to be 0, then a x equal to a y equal to a z they are all 3 are 0. On the other hand, e x, e y, and e x plus e y are linearly dependent because the third vector can be written as the sum of the first and the second. So, you have this relation which makes them linearly dependent, right.

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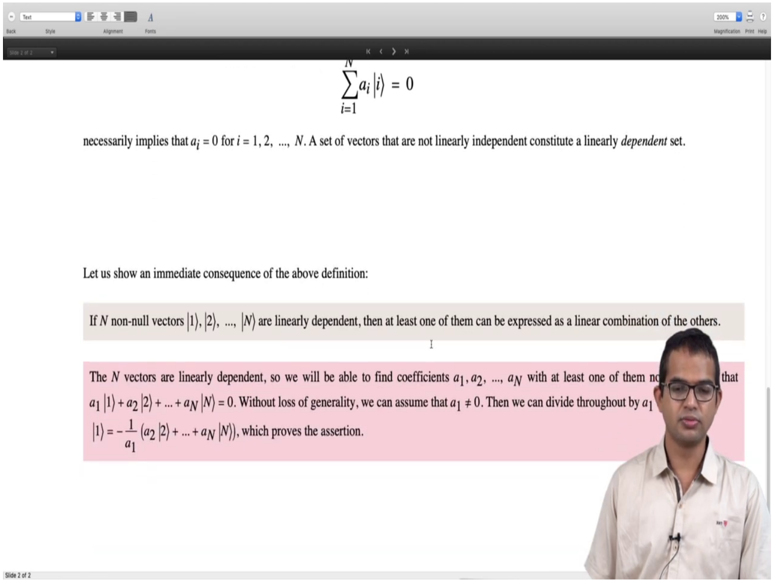

So, this notion can be extended to abstract linear vectors. So, the definition is like what you would expect. Consider some N vectors 1, 2 all the way up to N, they are linearly independent if the relation summation over some you know some coefficients a i, i if this sum equal to 0 and necessarily implies that all these coefficients must be 0. The only way you can satisfy a relation like this is if all these coefficients are 0, right.

Then, it means that you have pulled out a set of vectors which are linearly independent. So, and if a set of vectors is not linearly independent then it becomes linearly dependent, right. So, that is how you define a linearly independent set.

Now, there is an immediate consequence of this which I will show and then we will stop with this lecture. So, let so, if there are N non-null vectors 1, 2, 3, so on all the way up to N and if they are linearly dependent, then at least one of them can be expressed as a linear combination of the others, right. I mean it is not a surprise once you have seen the argument, but look at the way one develops the argument systematically and just for that purpose, right this is instructive, right.

So, we go back to the definition of linear dependence, right. So, these vectors are linearly dependent. You will have N vectors. So, therefore, sure you will be able to find a set of coefficients a 1, a 2 all the way up to a N and such that at least one of them must be non-zero, right. If all of them are 0, then that is not linear dependence, right. If there is at least one of them which is non-zero such that a 1 1 plus a 2 2, so on all the way up to a N N is 0.

Now, we are given for sure that at least one of these coefficients is non-zero, and so without loss of generality we can assume that a 1 is non-zero, right. You call that which is non-zero as a 1 and that vector corresponding to v 1, right.

Then, we can divide throughout. So, the point is that if one of these coefficients is non-zero then it's legitimate to divide throughout by this coefficient a 1 and then you can express 1 as minus 1 over a 1 times all of this stuff on the right hand side. Of course, all the other coefficients are free to be 0. If they are 0 then this vector 1 itself will turn out to be 0, correct.

The point is that there is a division done with respect to the division by a 1, and a 1 is taken to be non-zero, right. And we are guaranteed that there is such a 1, right. And that is why there is at least one vector which can be represented as a linear combination of all the other vectors in that set, right. You can have more than 1 of course, right.

And so, anyway it is not a surprise when you have thought about this a little more and you know got the flavour of what linear independence means, what linear dependence means. But it is important to go over this systematically because we will use some of these properties, this notion very critically we will show up in our discussions ahead, right.

Thank you. That is all for this lecture.