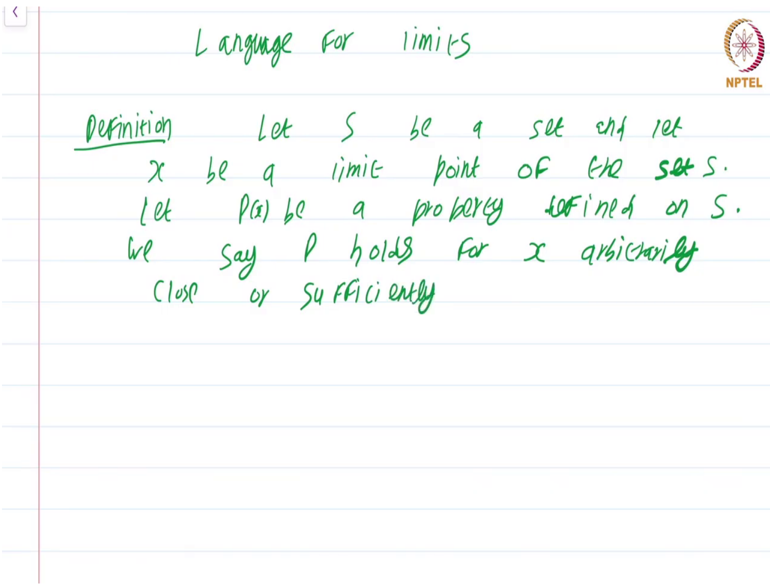
Real Analysis – I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 16.1
Language for Limits

(Refer Slide Time: 00:14)

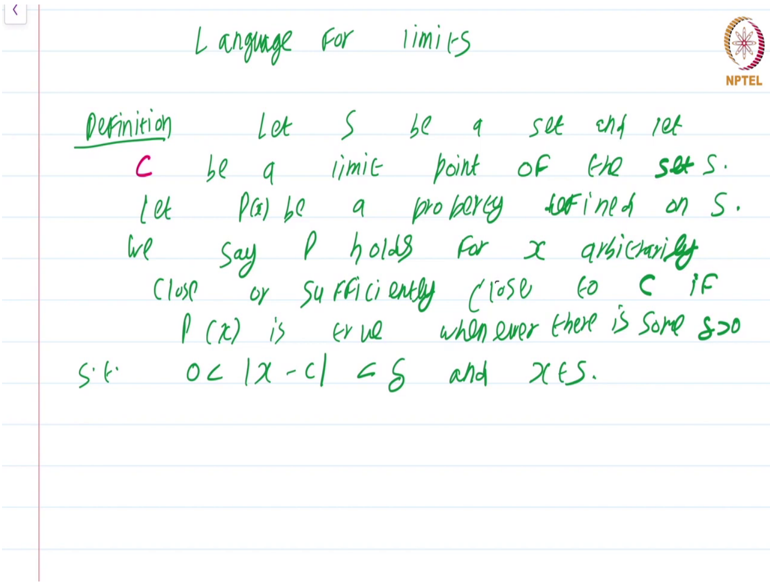


The notion of limits and continuity are complicated concepts. There are epsilons and deltas dangling everywhere, you have to choose δ appropriately, so that something happens so on and so forth.

We are now going to ease our burden by introducing a convenient language for speaking about limits. We will not use this language extensively because, this is after all a first course where you have to write everything properly and rigorously, but nevertheless and several instances in this course it will be very useful to have this language at our disposal.

So, just like language for limits, we talked about quantities getting close to each other as the number n became arbitrarily large right. So, we are going to introduce similar notions for limits. So, definition let S be a set and let x be a limit point of the set S. Let P be a property defined or rather let P(x) be a property defined on S; we say P holds for x arbitrarily close or sufficiently close.

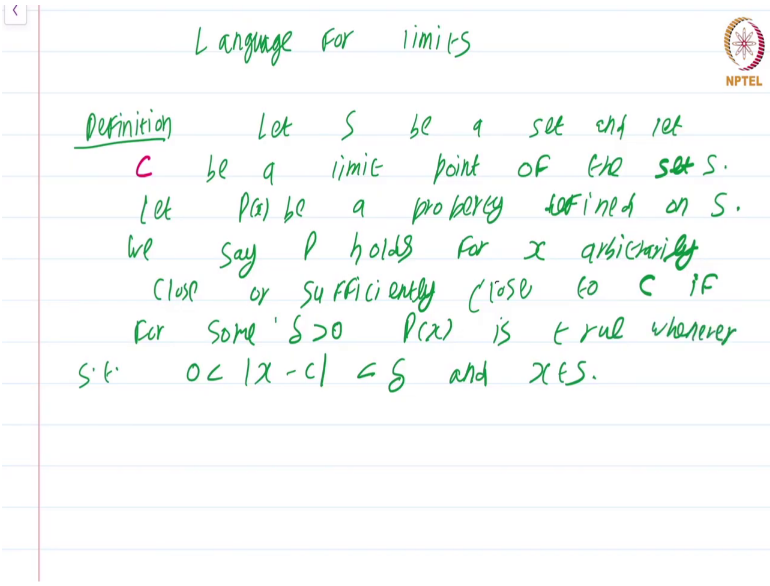
(Refer Slide Time: 02:22)



So, best to call this limit point something else so, that there is no confusion let us say ‘c’ be a limit point, close or sufficiently close to c. If P(x) is true, whenever there is some $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in S$.

So, let us revisit this: you have a set S and you are choosing a limit point C of this set S, you have a property P(x) defined on S, we say P holds for x arbitrarily close or sufficiently close to c if P(x) is true, whenever there is some $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in S$. So, to be 100 percent precise, if you do not mind let me just reorder this way I am writing.

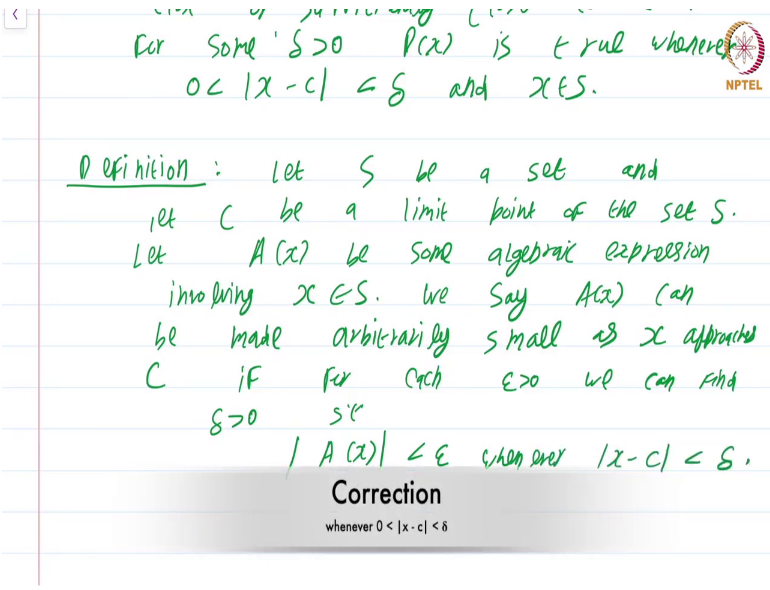
(Refer Slide Time: 03:46)



So, that things become clearer if for some $\delta > 0$, P(x) is true whenever $0 < |x - c| < \delta$ and $x \in S$. So, what this is saying is you can make the property be true, if you can find some δ such that P(x) is true, whenever x is sufficiently close to C and x is coming from S.

We do not allow x = c simply because this c may not even be in this set. And the way we write this is analogous to the way we define limits, it will be useful there.

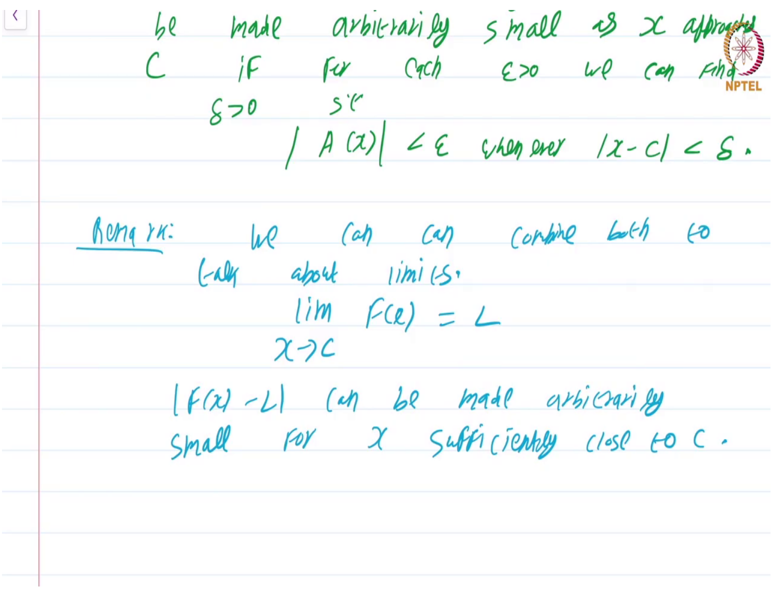
(Refer Slide Time: 04:40)



Now, next is another definition. Let S be a set and let c be a limit point of the set S, let A(x) be some algebraic expression, involving $x \in S$. We say A(x) can be made arbitrarily small, as x approaches c.

Now, this should be very very familiar if for each $\epsilon > 0$, we can find $\delta > 0$ such that $|A(x)| < \epsilon$, whenever $|x - c| < \delta$. So, this is just I mean this is just giving another meaning for the term limit. This is just saying that limit as x goes to c of A(x) goes to 0, we are just calling it can be made arbitrarily small.

(Refer Slide Time: 06:44)



Now, why is this first and second definition useful well

Remark: We can combine both these. We can combine both to talk about limits. How do we do this? Well, we can rephrase $\lim_{x \to c} F(x) = L$ as saying $|F(x) - L|$ can be made arbitrarily small for x sufficiently close to c. This is sort of combining the meanings of both the preceding definitions.

Instead of saying $\lim_{x \to c} F(x) = L$ just means that given any $\epsilon > 0$, there is a $\delta > 0$ blah blah blah. You can just say $|F(x) - L|$ can be made arbitrarily small for x sufficiently close to c.

This is a concise statement, but nevertheless with the preceding definitions it is a very precise statement and not only that this is a nice way of thinking about limits. You make x sufficiently close to c, $|F(x) - L|$ can be made arbitrarily small. We have translated our intuition into rigorous mathematics via $\epsilon - \delta$.

Now that we have processed, understood and made it part of our circulatory system, its best to now go back and try to use words that have more expressive meanings like arbitrarily small and sufficiently close. But, you should be very careful when you speak of this language.

Because, one when you use such language, there is always the tendency to delude yourself into thinking that you have understood what is going on or worse right proofs that seems to make perfect sense, but when you translate to rigorous mathematics the proof breaks down. So, in this course I will minimize using such language, but the literature uses such language all the time ok, it uses such language all the time.

So, it is best to be familiar with what this means. So, this is a convenient language for talking about limits. We can also talk about infinite limits that will be the content of another module, where we will be talking about arbitrarily large instead of arbitrarily small.

This is a course on Real Analysis and you have just watched the module on a Language for Limit.