Real Analysis - I
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Lecture – 15.7
Operation on Continuous Functions

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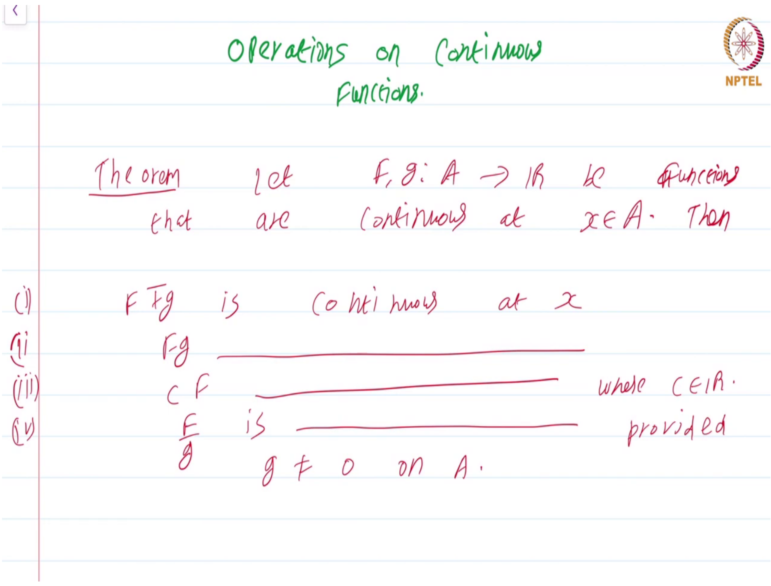

Because continuous functions and limits of functions are so intimately tied together, and functional limits are so intimately tied together with limits of sequences; the following theorem does not even deserve a proof.

Theorem: Let $F, g : A \longrightarrow \mathbb{R}$ be functions that are continuous at $x \in A$. Then

(i) $F \pm g$ is continuous at x.

(ii) Fg is continuous at x.

(iii) cF is continuous at x, where c is a real number.

(iv) $\frac{F}{g}$ is continuous at x provided $g \neq 0$ on A.

Of course, the last thing can be finished and a better version be made; but I am not going to bother doing it. So, this theorem just follows immediately from all the theory that we have so far developed.

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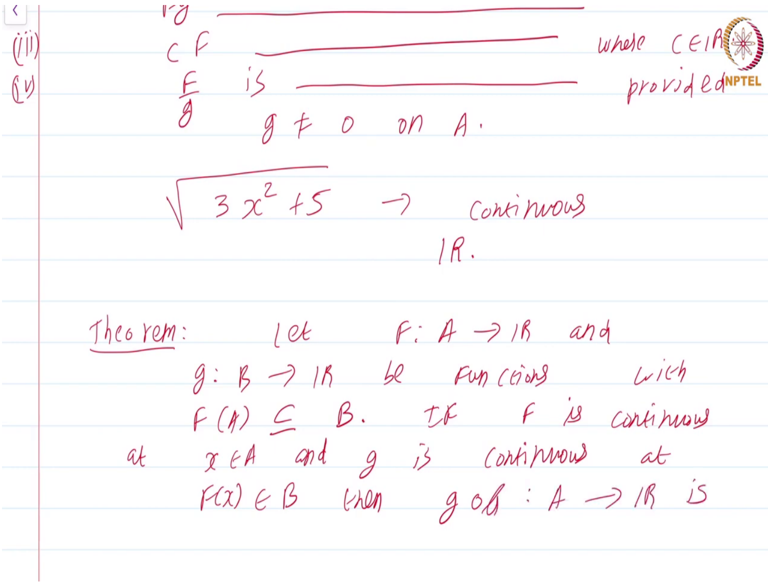

Now, suppose I were to ask you, look at the function $\sqrt{3x^2+5}$. Is this function continuous? First of all what is the domain of the function? The domain of the function is the whole of the real numbers, because there is an x^2 term inside the square root that will never be 0, it will be well defined.

We already know that because of this previous algebraic theorem, where sums, products etc.. of continuous functions is continuous; $3x^2+5$ will be continuous. And we also know that square root will be continuous; because that is what we proved in an earlier module.

But we cannot combine these two in any simple way to show that root of $3x^2+5$ is continuous just by using this algebraic theorem. Well, no problem; I will just prove what is needed.

Theorem: Let $F: A \longrightarrow \mathbb{R}$ and $g: B \longrightarrow \mathbb{R}$ be functions with $F(A) \subset B$. If F is continuous at $x \in A$ and g is continuous at $F(x) \in B$; then $g \circ F: A \longrightarrow \mathbb{R}$ is continuous at the point x,.

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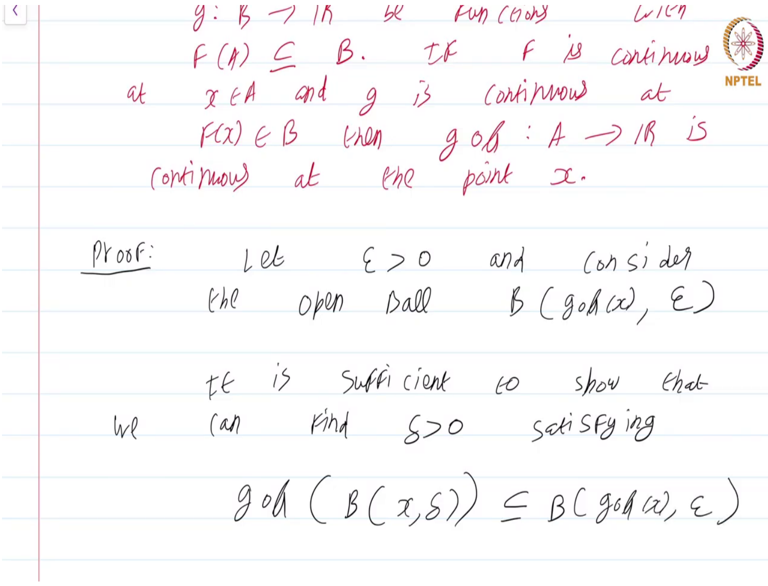

Proof: This proof is best done using open balls around the point that we are interested in. Let $\epsilon > 0$ and consider the open ball $B(g(x), \epsilon)$.

Now, by the various characterizations of continuity that we have seen along with properties of limits; it is sufficient to show that we can find $\delta > 0$ satisfying $g(B(x, \delta))$ is fully contained in $B(g(x), \epsilon)$. If we can do this, then we are done. How does one do this?

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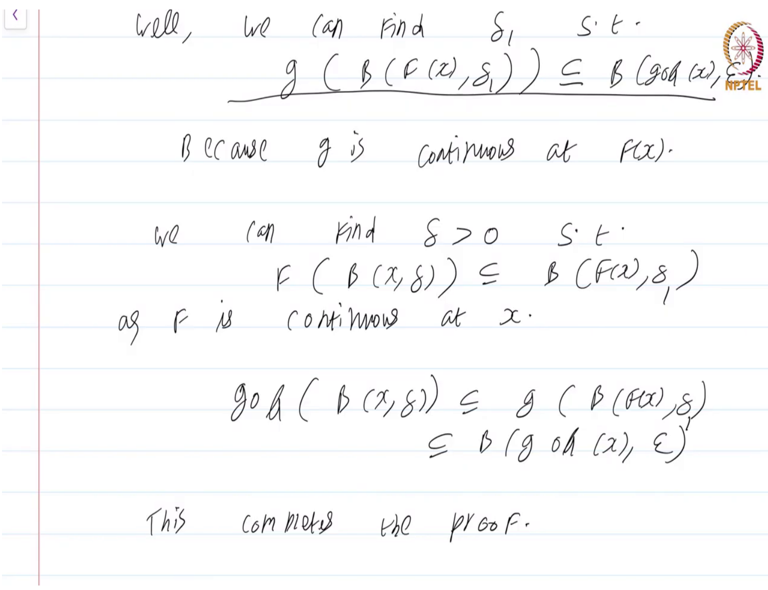

Well, we can find δ such that $g(B(x,\delta)) \subset B(g(x),\epsilon)$. Why can we do this? Because g is continuous at F(x), right.

Now, we can find $\delta > 0$ such that $F(B(x,\delta))$ is fully contained in $B(F(x),\epsilon)$, because F is continuous at x. Putting 2 and 2 together, we get 4, which is what we want $g(B(x,\delta))$ is fully contained in $g(B(F(x),\delta)$, which is contained in $B(g(x),\epsilon)$.

So, this completes the proof. So, this was fairly easy when we used open balls. So, the moral of the story is, we always try to characterize a definition in as many different forms as possible. This is because given any situation, you need to use the right tool for the job; when all you have in your hand is a hammer, everything starts to look like a nail, but you would wish to have a saw, a screwdriver, a spanner and so on.

So, we have characterized continuity in several different ways, use the characterizations wisely. This is a course on Real Analysis and you have just watched the module on Operations on Continuous Functions.