Real Analysis – I
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Lecture – 11.3
Tests for Convergence

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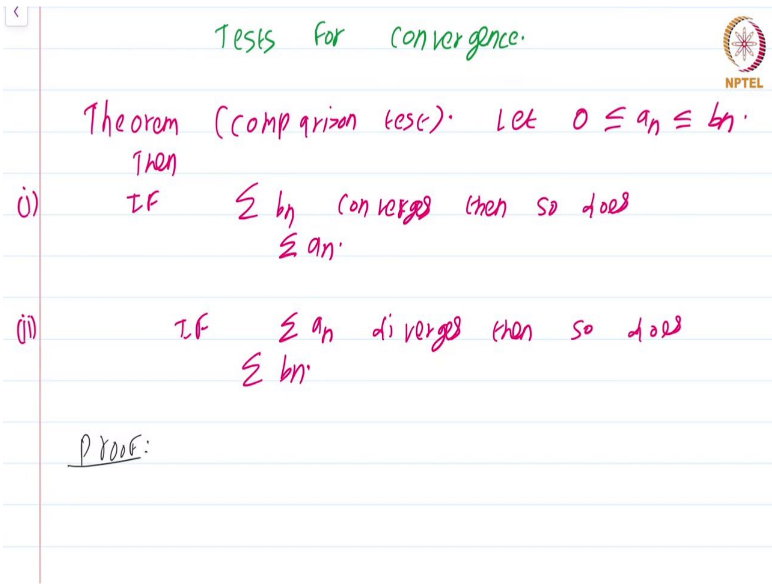

So far the only way of testing whether a series is convergent or not is to apply on a case by case basis some of the simpler theorems that we have proved like Cauchy's convergence criteria or monotone convergence theorem or Cauchy condensation test. Now, I am going to give a broad class of tests that are usually sufficient to deal with many different types of series that arise in practical use.

So, the heart of all these is the comparison test;

Theorem: (Comparison test) Let $0 \leq a_n \leq b_n$, then

(i) If $\sum b_n$ converges then so does $\sum a_n$.

(ii) If $\sum a_n$ diverges then so does $\sum b_n$.

The proof of part ii is fairly easy, so I will just prove part one.

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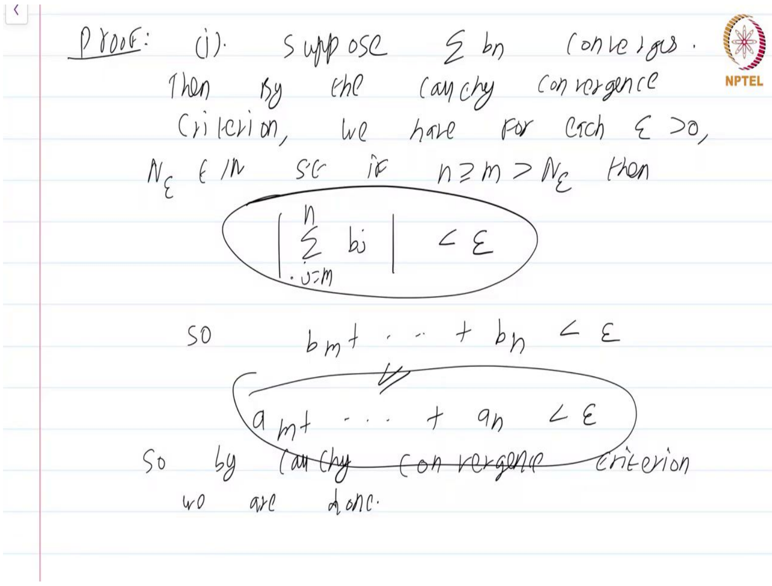

Suppose, $\sum b_n$ converges then by the Cauchy convergence criterion we have for each $\epsilon > 0,\ N_\epsilon \in \mathbb{N}$ such that if $n \geq m > N_\epsilon$ then $|\sum_{j=m}^{n} b_j| < \epsilon$, but these b_j 's are all positive quantities, that is our assumption.

So, what we get is $b_m + ... + b_n < \epsilon$. In other words, $a_m + ... + a_n < \epsilon$ because this is less than or equal to this; that is our hypothesis $0 \leq a_n \leq b_n$. So, by Cauchy convergence criteria again, remember the Cauchy convergence criterion is an if and only if condition. If a series is convergent then this must happen, if this happens then the series is convergent.

So, by Cauchy convergence criterion we are done. So, we have proved the comparison test that allows us to prove that a series converges by comparing it with some other convergent series.

Note that to apply the comparison test; you must have a tool kit of several convergent series at your disposal to compare with. You will develop the toolkit in the next few modules and in the exercises of course. So, it is always good to know some collection of nice series which converges and use it in the comparison test.

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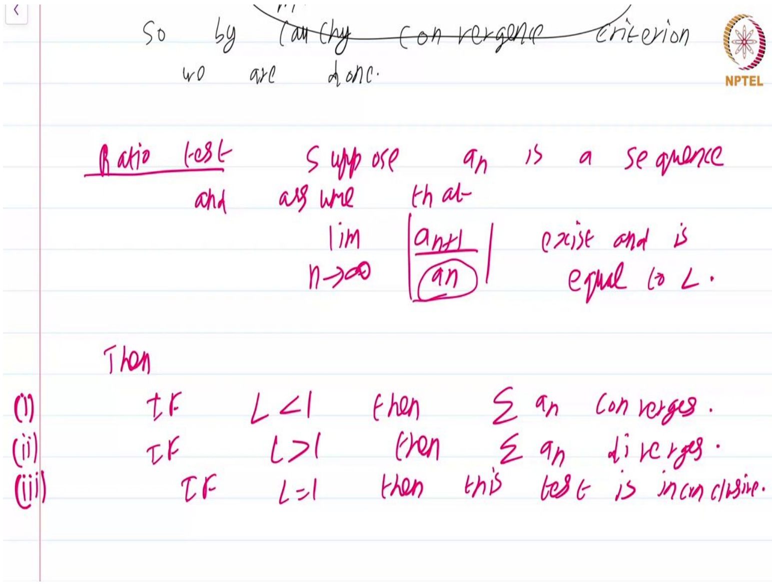

Let me illustrate by proving another popular test for convergence, this is called the ratio test. Let us prove this ratio test.

Suppose, a_n is a sequence and assume that $\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}|$ exists and is equal to L. Then

(i) If L < 1 then $\sum a_n$ converges.

(ii) If L > 1 then $\sum a_n$ diverges.

(iii) If L = 1 then this test is inconclusive.

Suppose, you take the limit of the successive ratio of absolute values of the sequence. In particular the denominator should be non-zero otherwise it does not make sense. I will make some remarks regarding this after we conclude the proof.

Suppose, the limit exists and is equal to L. If L < 1 then $\sum a_n$ converges. If L > 1 then $\sum a_n$ diverges, if L = 1 then this test is inconclusive. It does not give you any data. Let us deal with part three first.

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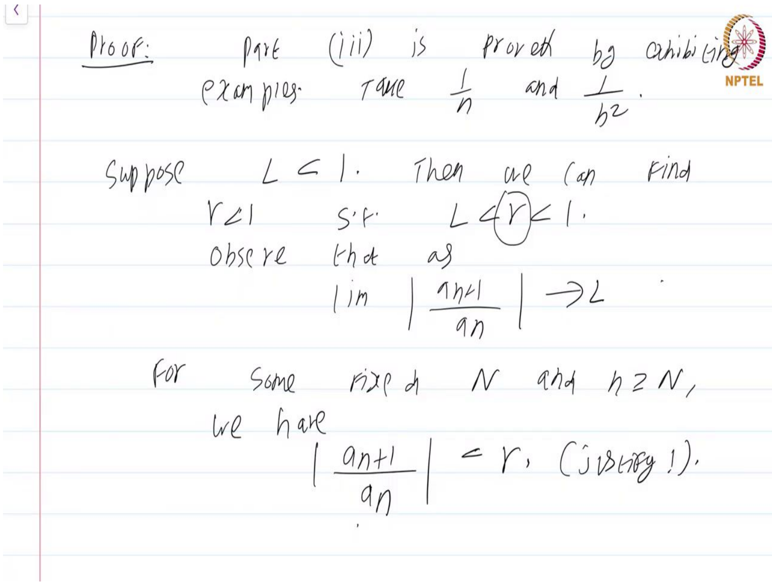

Let us deal with part (iii) first. Well, part (iii) is proved by exhibiting examples right. That is the only way you can prove part (iii). That means, you have to exhibit examples such that in one case the limit is 1, but the series converges. In the other case limit is 1, but the series diverges by exhibiting examples. Take $\frac{1}{n}$ and $\frac{1}{n^2}$ and I leave the details to you.

Now, suppose L < 1, then we can find r < 1 such that L < r < 1. You will understand in a moment why I have chosen this r this way and why I have used the notation r . Now, observe that as $\lim|\frac{a_{n+1}}{a_n}|$ converges to L, for some fixed N and $n \geq N$, we have $|\frac{a_{n+1}}{a_n}| < r$. Justify this.

What we have done is the following, we know that the sequence $|\frac{a_{n+1}}{a_n}|$ converges to L. And, we know that this L < 1. What we are doing is we are squeezing in this r in between L and 1.

By the $\epsilon - n$ definition of convergence for suitably large n, the ratio must be very close to L. In particular the ratio must be strictly less than r, whenever $n \geq N$.

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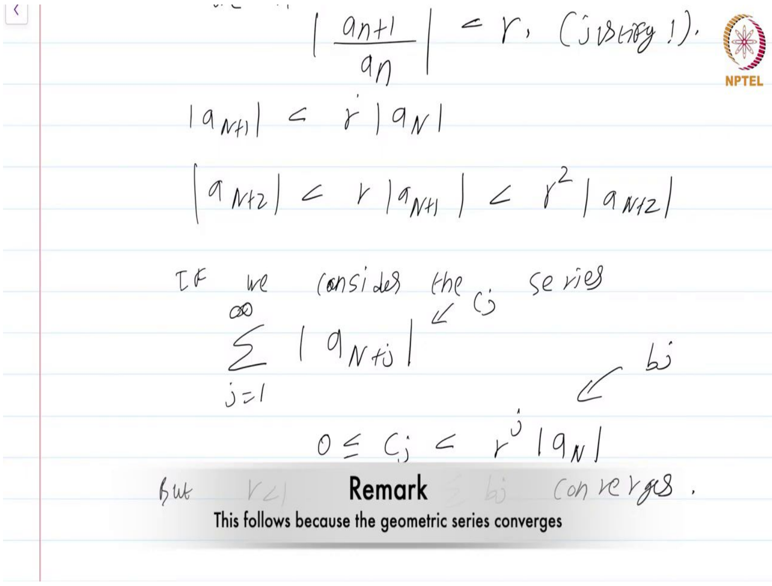

Now, how does this help us? Well, this shows that $|a_{N+1}| < r|a_N|$ right in particular. Similarly, $|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$. So, what will essentially happen is if we consider the series $\sum_{j=1}^{\infty} |a_{N+j}|$. If I consider this new series then I know that 0 is less than or equal to; call this new series let us say b_j.

Let me not call this b_j, let me call this c_j; $0 \leq c_j < r^j |a_N|$. As you can imagine we are now going to use the fact that the geometric series converges fine. So, how does this help? But, r < 1; call this b_j. So, $\sum b_j$ converges.

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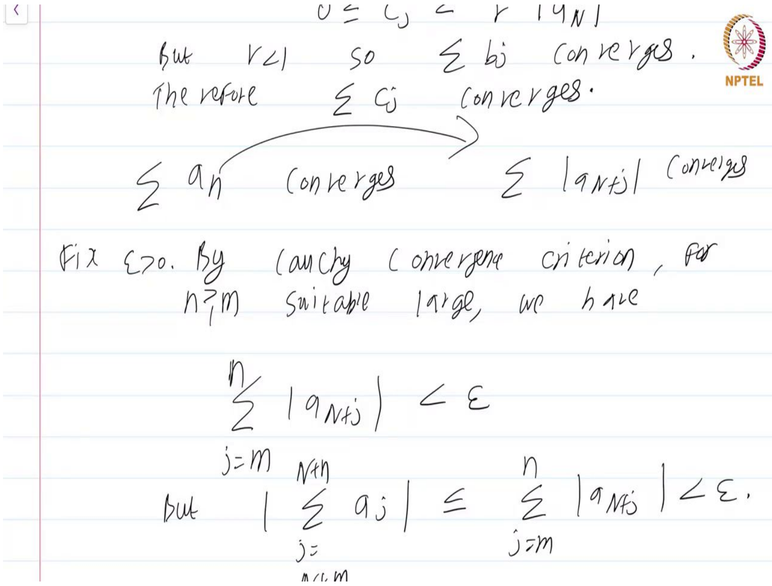

Therefore, $\sum c_j$ converges. This follows immediately from the comparison test. How does this help us? Well, observe that our original goal was to show that summation a n converges summation a n converges.

Now, what we have is $\sum |a_{N+j}|$ converges. This is what we have, but these both are intimately related to each other in the following way. By Cauchy convergence criterion for n,m suitably large we have $\sum_{j=m}^{n} |a_{N+j}| < \epsilon$.

So, I must fix $\epsilon > 0$. This is simply because $\sum |a_{N+j}|$ converges. But, $|\sum_{j=N+m}^{N+n} a_j| \leq \sum_{j=m}^{n} |a_{N+j}| < \epsilon$.

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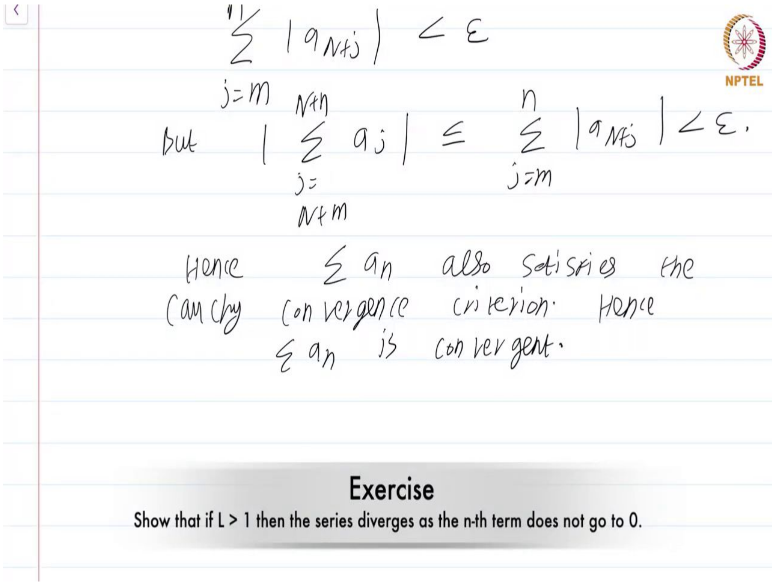

Hence, the series $\sum a_n$ also satisfies the Cauchy convergence criterion. Hence, hence $\sum a_n$ is convergent. So, this proof was a bit involved, but all I have done is apply the comparison test at this particular point. That is the only non trivial step in this whole proof; the rest of that is all just simple basic manipulation.

To somehow apply the comparison test, I have compared different series not the original series $\sum a_j$, but a different series to the geometric series $\sum r^j |a_N|$, which converges. And, by the comparison test we know that $\sum c_j$ converges and we have concluded that the series $\sum a_n$ converges .

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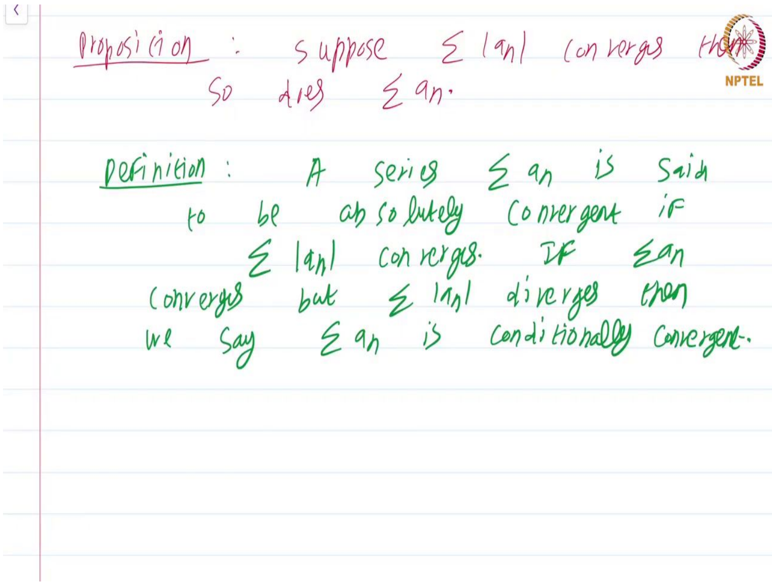

Now one portion of the proof that just involved an application of triangle inequality is a useful proposition. I am not going to prove this because we have already done it as part of the proof of the previous theorem.

Suppose $\sum |a_n|$ converges then so does $\sum a_n$.

We have done the proof of this as a part of the proof of the previous theorem.

So, this prompts the following definition.

A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges then then we say $\sum a_n$ is conditionally convergent.

Note very carefully that the root test because there are absolute values involved; this root test is actually a test for absolute convergence and not conditional convergence.

If you are going to conclude that a series converges using the root test you would in fact, be proving something stronger, you would be proving that the series is absolutely convergent. Now, one more remark I want to make regarding the proof of the root test, I mean the ratio test that we had done.

The proof of the ratio test we had done at one step involved creating this new series, creating this new series which is essentially just the later portion of the original series with an absolute value; essentially we call that the tail of the new series.

So, what we are essentially doing is we are ignoring the first few terms and just starting a new series from some particular point of a given series . Now, it must be 100 percent obvious to you that whether a series converges or not does not depend on the behavior of the first 10, 100, 1000 or 10,00,000 terms.

It depends only on the behavior of the sequence beyond something. And, this is amplified in the Cauchy convergence criterion where all you are required to show is that $\sum_{j=m}^{n} a_j < \epsilon$ for suitably large m and n.

Therefore, modifying the first few terms of a series is not really going to affect its convergence behavior. And, we have sort of seen this more rigorously, when I prove this in detail by considering the sequence $\sum c_j$ and comparing it with the original series right. So, keep in mind that all the convergence tests that I am stating can be modified to take into account these remarks, that the behavior of a series really depends only on its behavior of the tail.

So, for instance in the statement of the comparison test, I am saying that you require $0 \leq a_n \leq b_n$. By a similar proof to what we have done for the ratio test; you can show that you do not really need $0 \leq a_n \leq b_n$. You just require this inequality whenever n is suitably large. You do not require it for all values of n; you can modify the proof to see for yourself that what I am saying is true ok, excellent.

So, we have now defined absolute convergence and conditional convergence. In the next module we will see a test for detecting conditional, but not absolute convergence and give several examples.

This is a course on real analysis and you have just watched the module on tests for convergence.