Real Analysis - I

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Lecture - 11.2
Cauchy Test

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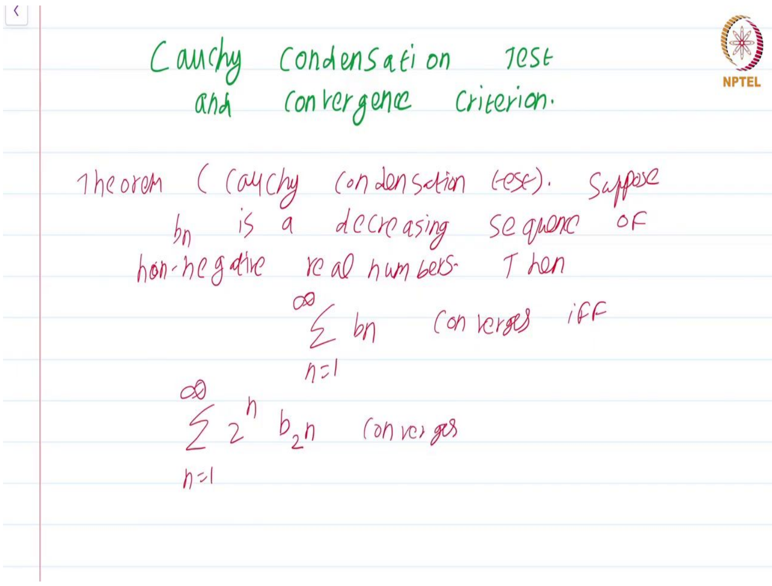

Motivated by the two examples, we have seen in the previous module that $\sum \frac{1}{n^2}$ converges and $\sum \frac{1}{n}$ diverges. Let us formulate and prove this theorem which is the famous Cauchy condensation test.

The test is as follows.

Suppose, b_n is a decreasing sequence of non-negative real numbers then, $\sum_{n=1}^{\infty} b_n$ converges if and only if, $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges.

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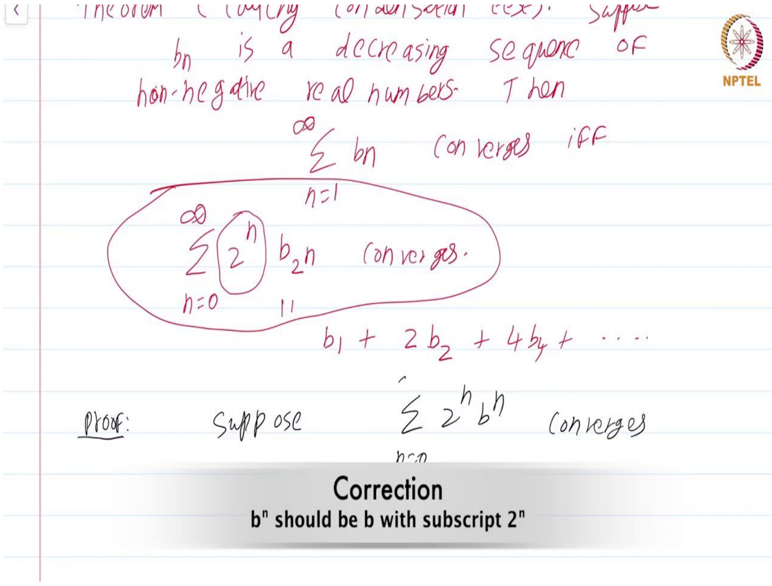

So, this is $b_1 + 2b_2 + 4b_4 + \ldots..$. So, the convergence of a series of non-negative terms that are decreasing is determined by the convergence of this new series, which actually has terms that are sparsely coming from b_n; but to account for the sparsity, you are multiplying by a exponential.

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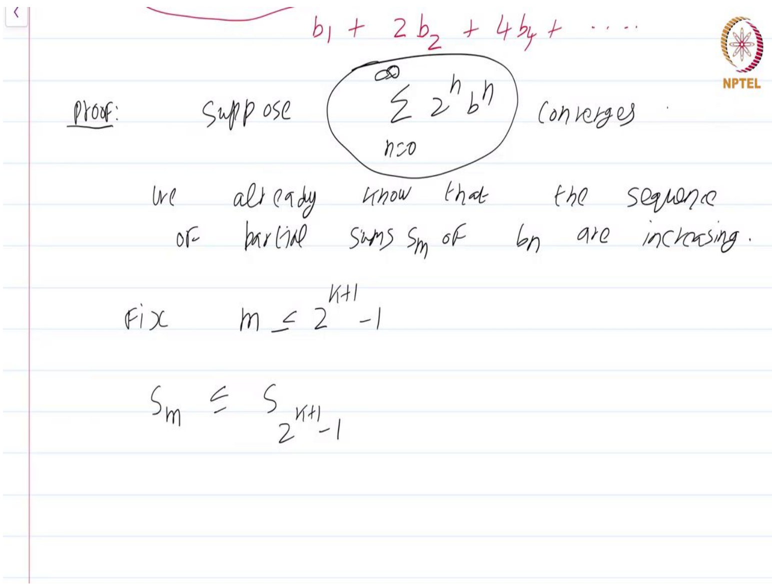

Let us prove this.

Proof: Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$. Now, what I do is, we already know that the sequence of partial sums of b_n are increasing right. Simply because the (b_n) is non-negative because of that the sequence of partial terms will be increasing. So, what I am going to do now is I am going to determine what S_m is using the fact that this series converges, $\sum 2^n b_{2^n}$.

How do I do that? Well, fix $m \leq 2^{k+1} - 1$, fix m like this. Then, the sequence of partial sums $S_m \leq S_{2^{k+1}-1}$.

The -1 is still there in the subscript. This is simply because as I just remark the sequence of terms $b_n,\ b_n \geq 0$. Therefore, this sequence of partial terms will be increasing fine.

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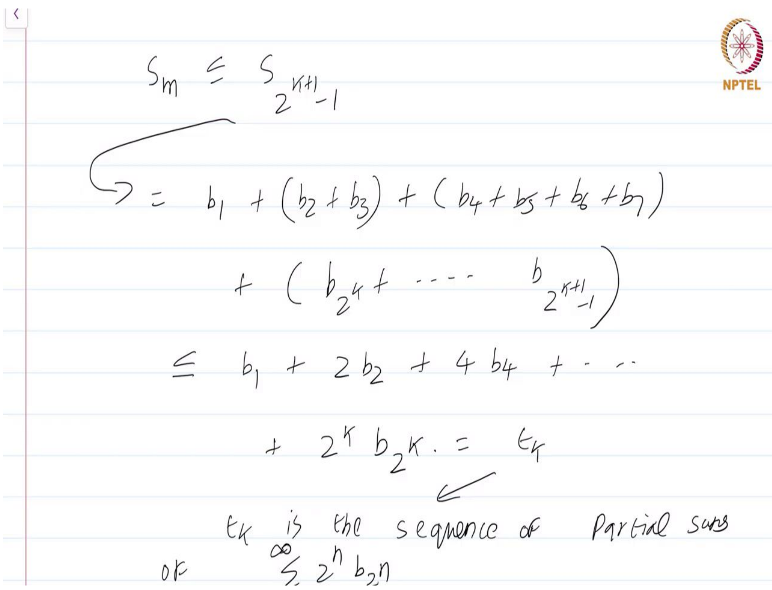

So, how does this help? Well, we pull the same trick, which is very very similar to what we have already seen in showing that $\sum \frac{1}{n^2}$ is convergent. This is nothing but $b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + . + (b_{2^k} + ... + b_{2^{k+1}-1})$.

Now, because this sequence is a decreasing sequence, b_n is a decreasing sequence, we have this is $\leq b_1 + 2b_2 + 4b_4 + ... + 2^k b_{2^k}$. Simply because this is a decreasing sequence. So, what have we got? We have got that this is nothing but t_k, where t_k is the sequence of partial sums partial sums of $\sum_{n=0}^{\infty} 2^n b_{2^n}$.

So, what we have actually established is that the sequence S_m is actually bounded above. Because we have established that if you take the nth term, where $m \leq 2^{k+1} - 1$. We have got that S_m is in fact less than or equal to t_k.

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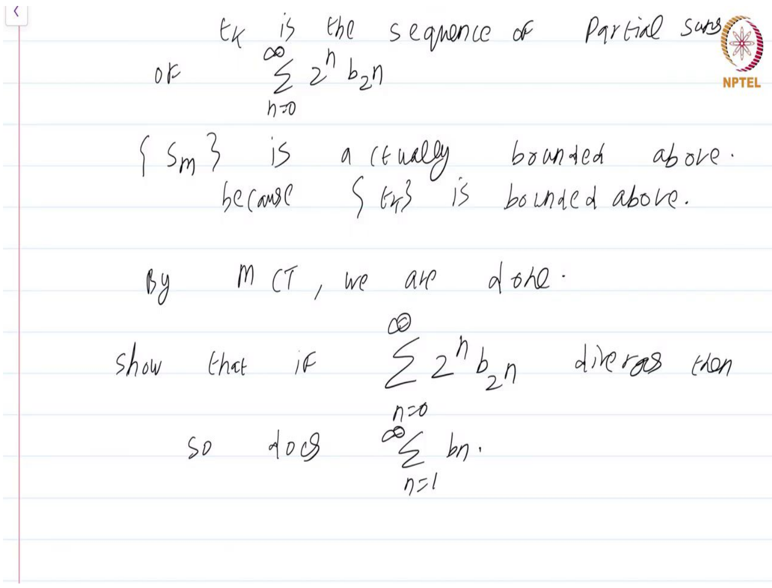

Therefore, the sequence of S_m is actually bounded above because this sequence t_k is bounded above. Why is the sequence t_k bounded above? Simply because the (t_k) converges, that is the hypothesis. Therefore, S_m is actually bounded above by the monotone convergence theorem, we are done.

So, now if you carefully look at the statement, it says if and only if, what we have shown is we have assumed that this converges. We have now shown that $\sum_{n=1}^{\infty} b_n$ converges. The other side is very very similar to showing that $\sum \frac{1}{n}$ is divergent.

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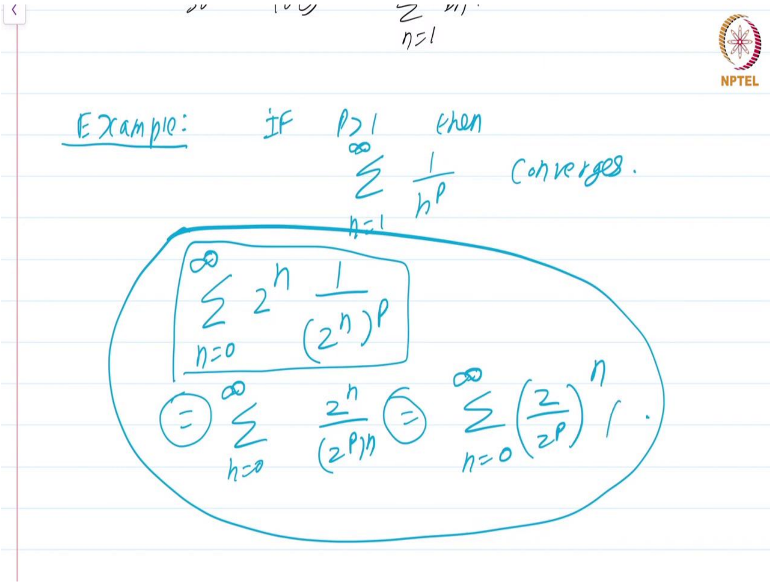

So, I am going to leave it to you, show that if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then, so does $\sum_{n=1}^{\infty} b_n$. This is exactly similar to the proof that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Now, immediately, we have an application of this Cauchy condensation test. Example, we already know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; we already know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Now, what happens in between? Well, if p > 1, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. So, p = 1 is sort of like the boundary; when p > 1, it definitely converges; when $p \leq 1$, it diverges. You will deal with the case '< 1' in the exercises . Suppose, you have $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. I am going to apply the Cauchy condensation test. Why? Because it is coming in that lecture.

Well, that is not the reason. You will see in a moment, why the Cauchy condensation test is very applicable. What we have to do is we have to consider this new series $\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p}$.

Now, why is this helpful? Well, what you can do is you can just pull this little trick. This is nothing but $\sum_{n=1}^{\infty} \frac{2^n}{(2^p)^n}$ which is nothing but $\sum_{n=1}^{\infty} (\frac{2}{(2^p)})^n$.

Now, one thing you should be careful about; this is a rather pedantic remark. The manipulations,\. I have done here about equalities. Note this is not really in a number unless it converges right.

A series you can treat it as a number only if it converges; otherwise, it is just the formal expression. Please recall that our definition of series is a formal expression. What I mean by $=$ here is that of logical identity. All I have done is manipulated what b_n is.

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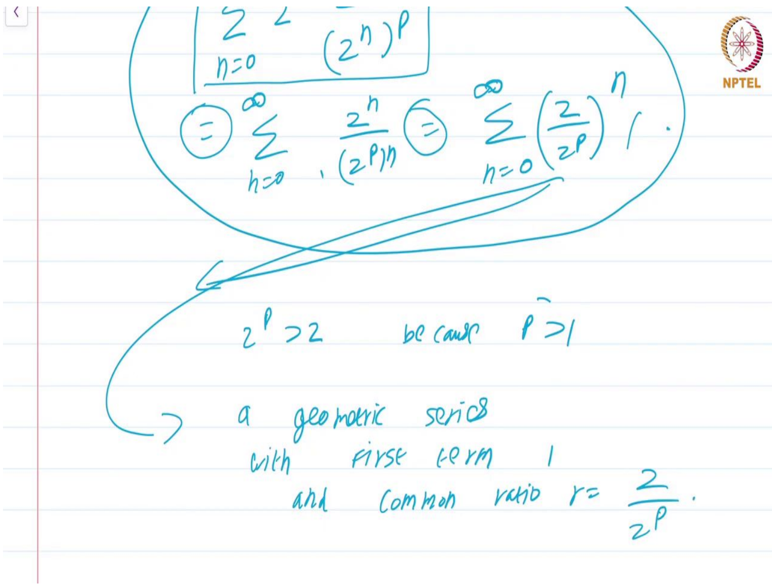

So, I am just asserting that this series formally is same as this series. I am not asserting anything about convergence and such things. Now, it will turn out that because these three series are logically identical. I have just manipulated the form of the expression b_n. If one of these series converges, then all of these converges and the number is going to be the same. The number to which it is converging is all going to be the same. This is a slightly pedantic remark .

Now, coming back, we have $(\frac{2}{2^p})^n$. Why is this helpful? Well, $2^p > 2$ because p > 1. So, because of that this just becomes a geometric series.

It becomes a geometric series with first term 1 because n = 0 and common ratio $r = \frac{2}{2^p}$. Now, we have already seen that if r < 1, this series converges. Hence, we are done.

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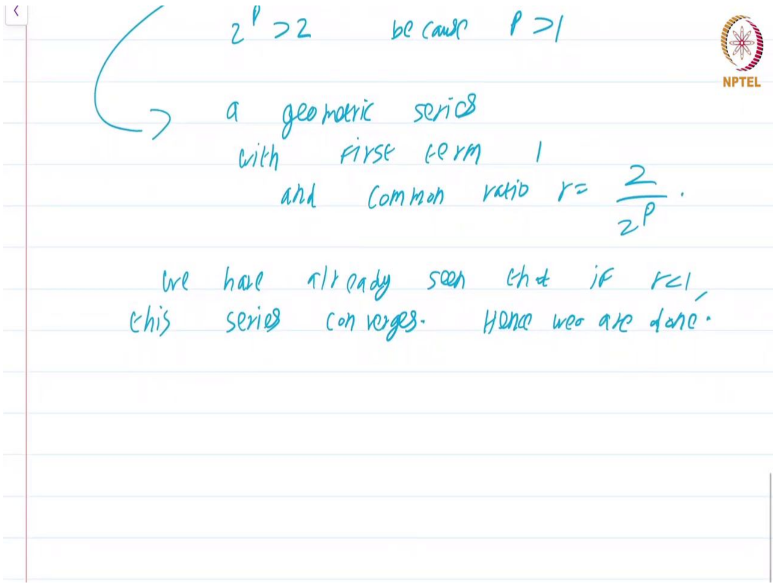

So, we have established that the series $\sum \frac{1}{n^p}$ converges if p > 1. This is an application of the Cauchy condensation test. Now, coming to the second topic of this module that is Cauchy’s convergence criterion.

This is a very very easy thing; simply because we have already applied this several times when studying sequences. So, suppose (b_n) is a sequence. So, let me just use consistent notation. I have chosen to avoid putting the parenthesis whenever possible.

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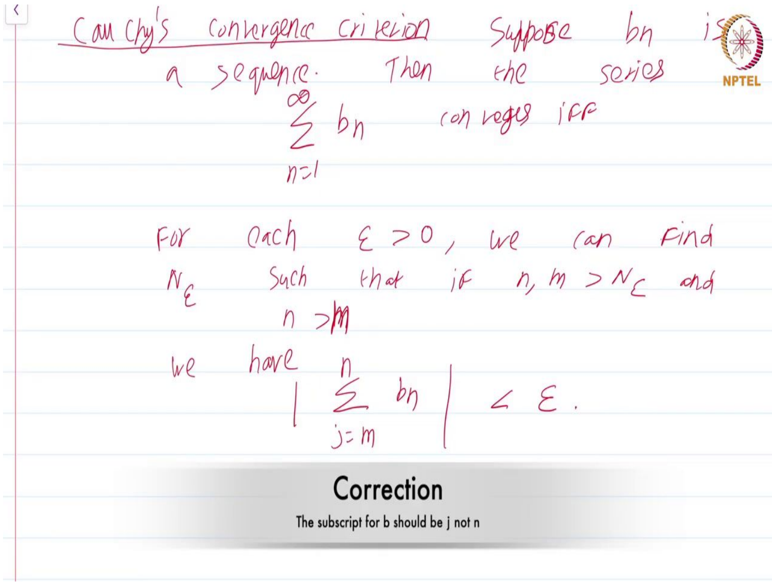

So, suppose (b_n) is a sequence, and let me just leave it as it is. Suppose, b_n is a sequence; then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if for each $\epsilon > 0$, we can find N_ϵ such that if $n, m > N_\epsilon$ and n > m, we have $\sum_{j=m}^{n} b_j < \epsilon$.

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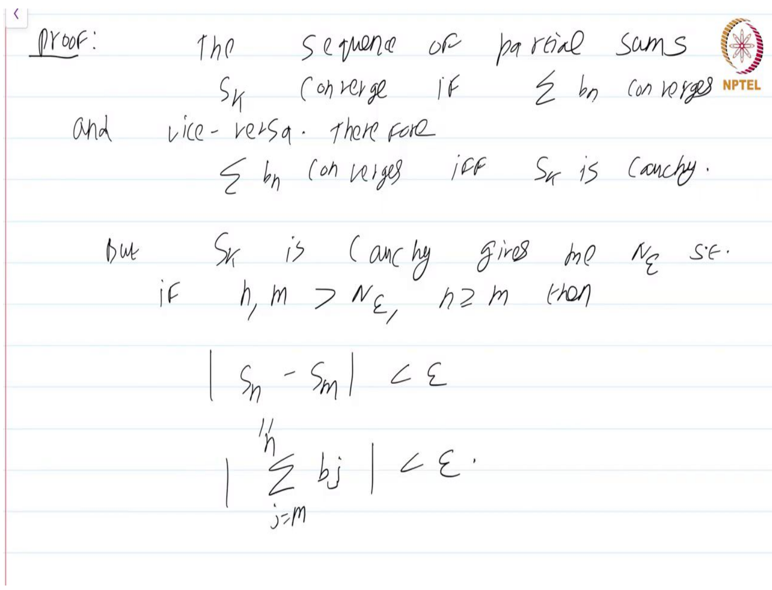

So, this might seem a bit complicated; but once you see the proof, you will just facepalm yourself and think, its obvious proof. Well, the sequence of partial sums; just one moment, let me just for increased applicability of this result put $n \geq m$. Really the sum still makes sense. The sequence of partial sums say S_k . S_k converge and therefore, sequence I should not write this. Converge if $\sum b_n$ converges right and vice versa and vice versa.

So, the sequence of partial sums converge if $\sum b_n$ converges and vice versa, that is just the usual Cauchy criterion for convergence of sequences. Sorry, that is just the definition; sorry about that. Therefore, $\sum b_n$converges if and only if S_k is Cauchy. This is where Cauchy convergence criterion comes into the picture, not the previous segment; sorry about the goof up.

So, $\sum b_n$ converges if and only if, S_k is Cauchy. But S_k is Cauchy gives me $N\epsilon$ such that if $n, m > N_\epsilon$ and $n \geq m$.

Then, $|S_n - S_m| < \epsilon$ right. That is what Cauchy convergence criteria is. But this is just the sum that we are interested in $\sum_{j=m}^{n} b_j$. This is exactly that sum . So, this is just I do not know what to say; this is just a proof by translating what the hypothesis is saying.

So, there is nothing really happening you have just reduced it to the Cauchy convergence criterion for sequences. So, both directions follow immediately. Because you can just reverse the steps in the proof, we have written.

Hence, proved. We get an immediate corollary which is a very very useful criterion for determining that a series diverges.

Corollary: A series $\sum b_n$ that is convergent must have $b_n \to 0$.

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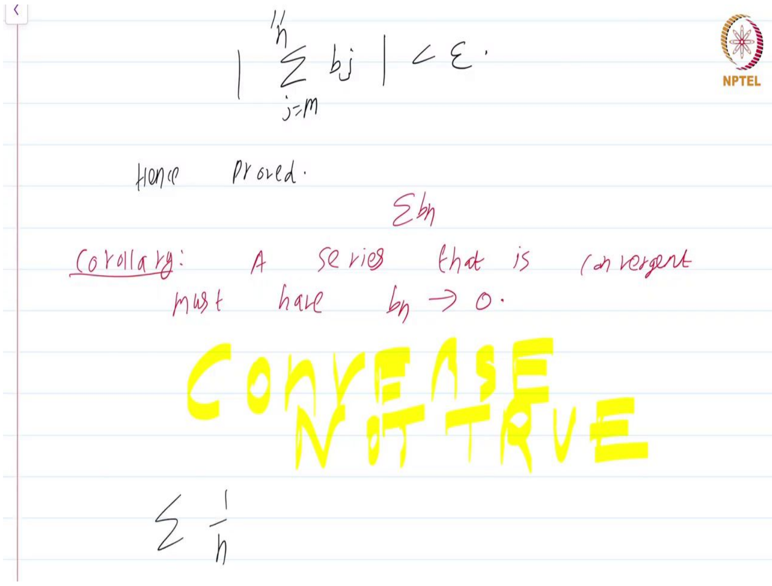

I am not even going to insult your intelligence by writing a proof. This just immediately follows from Cauchy's convergence criterion. So, the nth term of a convergent series must converge to 0 . So, the converse of this corollary is not true.

So, let me write that in bold because people make that mistake ok, converse ok. This is a very ugly color; not true. I am going to permanently imprint this in your head, converse is not true; do not make that mistake. If the nth term goes to 0, does not mean that the series converges; just take $\sum \frac{1}{n}$, that will give you the example .

So, that is it for today's module. This is a course on real analysis and you have just watched the module on the Cauchy condensation test and Cauchy's convergence criterion.