Introduction to Probabilistic Methods in PDE
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Revision of Conditional Expectation Part 1

In the last lecture, we have seen what is Random Variable.

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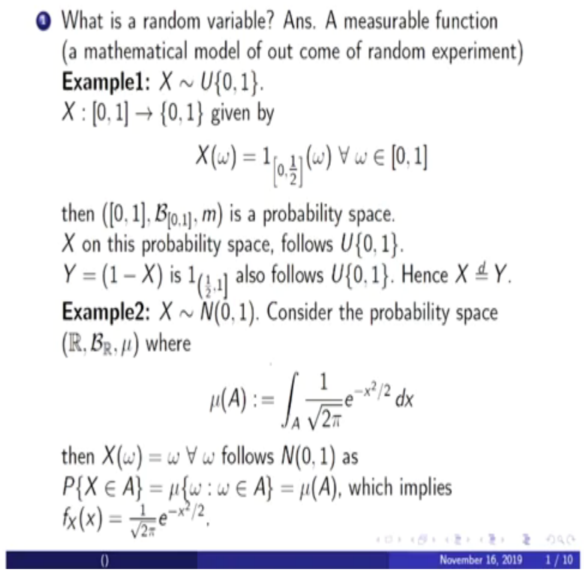

So, random variable is a measureable function for our course. So, we take mathematical model of random variable. We have seen three examples.

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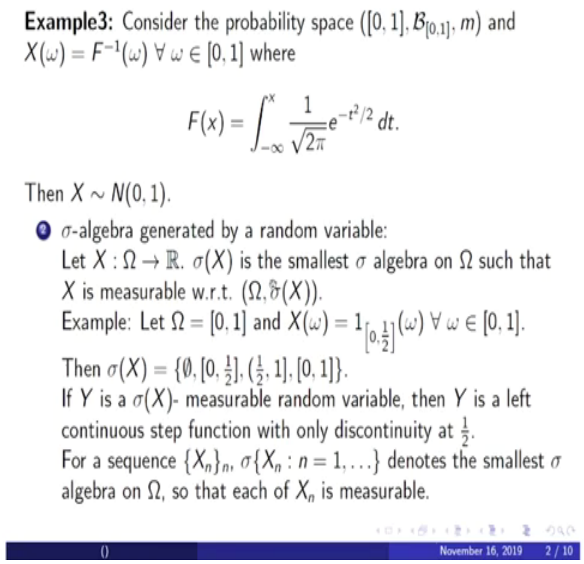

So, now today I would proceed further to talk about what is a Sigma Algebra generated by random variables and other issues. So, sigma algebra generated by random variable when we say that, we first have a map. So the map is X which is coming from Omega to R and then sigma X is the smallest sigma algebra on omega such that X is measurable with respect to the measurable space omega sigma X.

So, the basic reason of existence of that sigma algebra is that the collection of sigma algebra for under which X is measurable is non empty as the power set of Omega is already in that collection and then arbitrary intersection of sigma algebras is still a sigma algebra. So, from that one can get smallest algebra for which X is still measurable map. So, let us see a quick example.

If we take omega as closed interval 01 then, the following indicator function, the indicator function of 0 to half which we earlier have seen as an example of Bernoulli trial. So, for that the sigma algebra generated correspond this particular map would contain exactly four sets, the empty set, the full set and the set 0 to half and another is open half closed one.

So, how do you get it? You just take inverse of 1 with respect to the map X and then you get close 0 to half. So, one is a singleton set and singleton set is a bored set. So, inverse image of that should be measurable if my function X is measurable. So, I must have closer to half inside the sigma algebra under which X should be measurable and that justifies the compliment that is half open one closed should also be inside that collection.

So, next we also like to recall that if Y is a random variable which is measurable with respect to sigma X then Y is constant on these intervals 0 to half and half to 1 or in other words you can say that Y is left continuous step function with only discontinuity at half. So, here we cannot just derive any particular value of Y. However, we just can derive this maths that Y should we constant on these sub intervals.

So, any such function Y would be measurable with respect to sigma X. Now, we see what happens if we have a collection of random variables instead of a single random variable for that. So, we consider a sequence Xn of random variables, here sigma of sigma Xn denotes the smallest sigma algebra on omega so, that each of Xn is measurable. One might not need to consider all the countable collection, one can take arbitrary…I mean large collection also.

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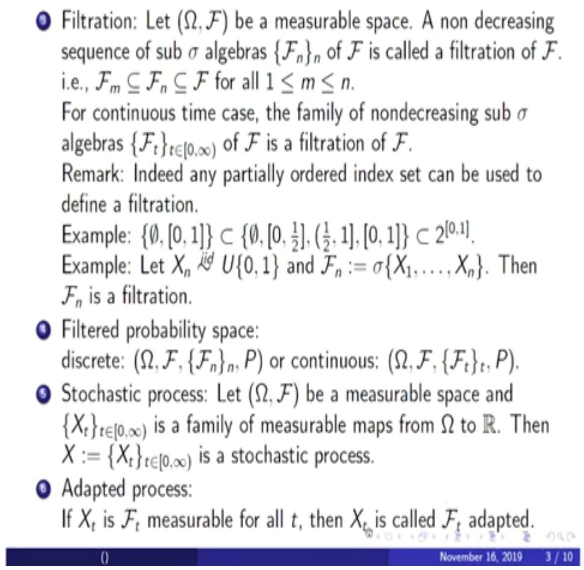

So, here we introduce what we call as filtration, we need this term throughout the course. So, this is an important concept. Let omega F be a measurable space in non decreasing sequence of sub sigma algebra of F, we write down F subscript n and this is a family you know countable family of sigma algebras is called a filtration of F, if these are non-decreasing. So, like Fm is a sub sigma algebra fn for all m less than or equals to m.

Now, this is a description for countably many sub sigma algebras, accountable filtration. However, for continuous time case one needs an uncountable family of sigma algebras. So, for continuous case the family of non-decreasing sub sigma algebras, F subscript t, here t we take from close interval to infinity of F is a infiltration of F.

So, we make a remark that indeed any partial order index set can be used to define a filtration. Because we are not using any algebraic structure of the index set correct, we just need that, I mean m less than equal to n etc. So, just order of the set P poset is sufficient.

Now, let us see some example of increasing family of sigma algebra. So, here is a trivial example, that on the left hand side you have only trivial sigma algebra let say closed intervals 0, 1 and on the right hand side you have the power set of close 0, 1, set of all subsets of close 0, 1 and in between the sigma algebra what have I have seen in the last slide appears that which contest empty set and these half interval and the remaining and the full.

Now, that this is not a infinite sequence of sigma algebras, next example we see that, we take independent and identically distributed random variables Xn which are Bernoulli. Bernoulli in the sense that they are uniformly distributed on by binary values 0 and 1. So, that means it takes value 0 with probability half and one probability half. Then, if we define F subscript n as sigma algebra generated by X 1, X 2 etc up to n. Then we are going to get an increasing family of sigma algebras because, you know, every time I am going to add one and one more random variables and then we are requiring Fn such that all of X1, X2, Xn should be measurable to do that.

So you are asking more and more functions to be measurable which is the sigma algebra. So, the sigma algebra is non-decreasing. We cannot say it is strictly increasing because one can of course choose you know, for some other cases all X are same, but here one cannot because it is independent. So, all X are not same so, one would actually get increasing family of sigma algebra is in this case. So, these are examples, we call this Fn filtration due to the above definition.

Now, we introduce what we call filter probability space. So, here we denote, this combination, where Omega is a non empty set, F is a sigma algebra and Fn is filtration, that means the family of sub sigma algebras and P is the measure, which is defined on (Omega F). So, this whole collection I mean to together we are going to call as a filter probability space, when we are dealing with continuous time case then we would rather consider omega F, F t. So, when I write down t small t here, we would always mean that we are talking about close 0 to infinity as the index set.

So, most of the time we are not going to write down that explicitly. Now, onward and soon you would actually talk everything in this setting only, not countable setting. So, countable setting is discussed here just to get the first intuitive understanding.

Now, stochastic process. What is stochastic process? It is a family of random variables. So, given a measurable space Omega F if we have a family of measurable maps that means random variable from Omega to R then X is a stochastic process. So, here we are restricting ourselves to only real value random variables. There is no problem, one can actually take X as a stochastic process, Y as a stochastic process and can take ordered pairs, (X, Y).

That would be a again stochastic process in two dimensional space R2. And then one can add one by one etc. So, then one can get actually using the same setting with no additional difficulty, stochastic process on any finite dimensional Euclidean space.

However, that is not only the end of this topic one can actually take stochastic process on higher dimensional spaces also. But for our purpose, it is sufficient to talk about only finite dimensional space so, stochastic process on the final dimensional space and therefore, all my definitions would be only for R, real valued to define and whenever we need Euclidean space values, just stochastic process that would be understood immediately.

Now, adapted process. So, if Xt is Ft measurable for all t then, Xt is called Ft adapted. So, adapted process. So, this name we should remember because we are going to use this heavily throughout the course. I mean when you say Xt is Ft measurable for all t then Xt is called Ft adapted, I mean here we mean that not the particular random variable Xt at time t but we mean that the process X throughout the timeline and Ft is not a particular sigma algebra but the collection of sigma algebra, the whole filtration. So, the better notation is actually putting a curly bracket on both the sides here in here.

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- Usual filtration due to a stochastic process:
  Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_t : \Omega \to \mathbb{R}$ for all $t \geq 0$. Define $\mathcal{G}_t := \sigma\{X_u : 0 \leq u \leq t\}$. Let $\mathcal{N}$ be the set of all null sets, i.e. subset of some zero measure subsets of $\mathcal{G}_t$. Then $\mathcal{F}_t := \sigma(\mathcal{N} \bigcup \bigcap_{u>t} \mathcal{G}_u)$, is called the usual filtration of $X := \{X_t\}_t$.
- Modification of a stochastic process:
  $P(X_t = Y_t\ ) = 1 \forall\ t \in [0, \infty)$.
- X and Y are indistinguishable if $P(X_t = Y_t\ \forall\ t \in [0, \infty)) = 1$.

Example: $\Omega = [0, \infty), \mathcal{F} = \mathcal{B}_{[0,\infty)}$, P is atomless

$X_t(\omega) = \begin{cases} 1, & \text{if } t = \omega \\ 0, & \text{else} \end{cases}$ then $P\{\omega : t \mapsto X_t(\omega) \text{ is continuous}\} = 0$.

$Y_t(\omega) = 0\ \forall\ t\ \forall\ \omega$.

$P(\omega | X_t(\omega) = Y_t(\omega)) = 1\ \forall\ t$.

X and Y are modifications but are not indistinguishable.

Usual filtration due to a stochastic process is very important notion when nothing is mentioned. So, in a context when nothing is mentioned about the filtration just one is talking only about the random process, then a default filtration is always there. So, what is the default filtration? So, let us describe it. So, first default thing is that you just define Gt which is sigma algebra generated by Xu where u is from 0 to t, close 0 to t.

So, that Gt would of course form a filtration, however for some applications and for proving some interesting theorems, we need some more property of a filtration, what are these? One of these is right continuity and another thing is the completeness.

So, these are the notions which I am not presenting separately but in together manner. So, consider that n is a set of all null sets. What do I mean is that a subsets of 0 measure sets, says subset 0 measure subsets of the filtration Gt. So, actually subsets of not Gt but for F, all null subsets of F, then Ft defined by sigma algebra generated by n and G u u greater than t and the intersection. What does it mean? That I have Gt, but I do not consider that, I consider all G u which is where u is more than t.

So, all sub sigma algebras which is after Gt and take integration of all these. So, like not coming from the right hand side and as I have told that arbitrary intersection of sigma algebra is Sigma algebra. So, this intersection of G u u greater than t is itself is sigma algebra and if you change your t, increase your t that would also increase. So, it is a filtration. So, this is itself this part itself becomes a right continuous filtration but we also augment some more sets in N is a set of all null sets. Yes, any question?

Student: Like whole null set of F.

Professor: So, this is a mistake here, I should ever written F here. Otherwise it looks like n it depends on t. So, and then if you take union of sigma algebra with a set you might not, I mean set means with a collection of subsets you might not get a sigma algebra at the end. So we must talk about the sigma algebra generated by this collection. So then you are assured to have a family of sigma algebras which is increasing and which is right continuous and which is complete, complete in the sense that, that every subset of 0 measure set is measurable here.

So, this particular choice of filtration is canonical, and we call this Usual filtration of the given process X or we sometimes say that let Ft be the sigma algebra generated by X satisfying usual hypothesis. So, there are many different way to describe or to specify this filtration within the text. So, there are many definitions in this slide because this is the introductory part of the course and the aim is to have all those together at one place so that it is easy to refer.

So, there is a notion of modification of stochastic processes. So, if you imagine you have two stochastic process X and Y and they are such that for each and every t they are equal with probability 1. Xt is equal to Yt and that happens with probability 1 and that happens for all t then, what do you call? You would call that X and Y are modification of each other, one also say that X and Y are your version of each other, X is a version of Y, Y is the function of X, etc. So, this actually is an equivalence relation correct? If X is modification Y, Y is modification Z, then X is also modification of Z, which has (())(17:29), everything.

However, there is another stronger notion, which is that X and Y would be called indistinguishable if, X and Y the path wise they match with probability1, why path wise I am saying? Because this relation is true for all t. So, for every omega path, every sample point, the process would result in a full path, correct for every t and then that path of X and that another path coming from Y for the same event same, same sample they would coincide and that coincidence would be observed with probability 1.

Now, what is the difference between that? That can be explained with one small trivial example, consider Omega to be close 0 to infinity open and if borel sigma algebra on Omega P is atom less. So, that means for any given point in that P of that singleton is 0. So, there are plenty of such atom less measures. So, that much you just assume and then define one stochastic process in the following manner, for each omega, fix omega and then you are defining that how it moves as t grows.

So, it is 0 every time only when t is equal to omega, then its 1, because my omega is now time like domain, closer to infinity. So, I can always talk about it so, wherever my omega is chosen for capital omega and then Xt would be 0 before it is omega and when at omega it is 1 and then again it becomes 0.

So, at the end, what are you going to get? You are going to get a stochastic process which is 0 for every time on the atomic rate is 1. So, it is discontinuous function, the path is discontinuous. So, if you ask that what is the probability that you are going to see that this t to Xt, this map is continuous? That probability is 0, because no (())(20:02) is like that.

However, if we choose another process Y were you do not have the perturbation, it is 0 all the time, and then you ask for which omega Xt and Yt are same. So if fix one t, anytime t and ask for what omega is, you know, this Xt and Yt would be the same, we would see for all omega is which is not equal to t both are 0. So they coincide, so this is true for all except t. So the measure would be 1 minus measure of singleton t, but singleton t has 0 measure because P is atom less. So, this probability is 1 for all t.

So, this proves that this example of X and Y. So, these are modification to each other. However, if we ask that path wise are the same? No, path wise they are never same. So, they are surely not indistinguishable. So, this example actually you know illustrates the difference between these two notions.

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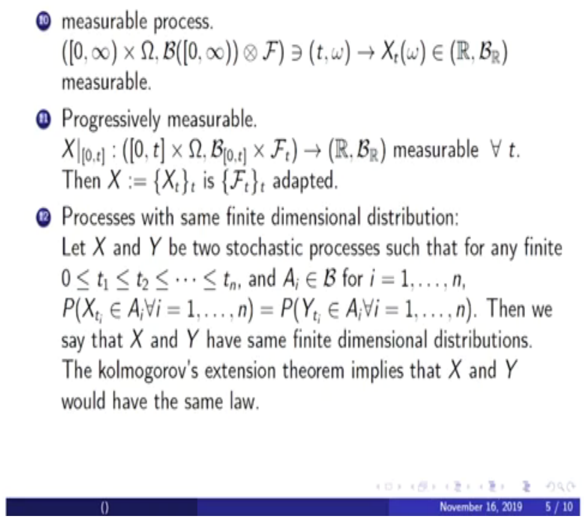

We are talking about now measurability of stochastic processes. So, what do we mean by a stochastic processes is measurable? So, here we must notice it, this is a very important thing that a stochastic process is actually a function of two variables t and omega, t is time and omega is the sample point. Now, this t omega is coming from, t is coming from this close 0 to infinity (())(21:52) omega smaller is coming from capital omega. So, therefore, this order pair is coming from this (())(21:59) is product of this thing and on this set one can consider a sigma algebra.

I need to fix the sigma algebra to talk about measurability, correct? So, that sigma algebra, we are taking that on 0 to infinity, here say borel sigma algebra and on omega we have already default choice, default choice of F and then if we take Cartesian product. So Cartesian product of sigma algebra is not a sigma algebra, correct? However, there is a notion called product sigma algebra, one can come up with the products in algebra. So, there are many different way to do that actually.

However, we are going to take the approach for which this becomes saturated or the Cartesian extension way, that I mean in a manner so that this algebra would not have any non measurable null set. I am going to include all subsets of null sets and on the range set I have R BR the borel sigma algebra on R. So, this is the meaning of measurable process if X satisfies this condition that when in the domain we have this sigma algebra and on the Codomain (())(23:27) borel sigma algebra, still this map is measurable, then we call X is a measurable process. However, there is other notions of measurability properties of X is called progressively measurable.

So, what is progressively measurable? That here we really take care of at time t, over time t how that you know, Xt is having measurability because in the first case on the left hand side you just had the sigma algebra together where time t does not play any role, here we take that process X restrict it to interval 0 to t. So when you restrict to 0 to t, then it is a process on 0 to t.

So, we think the t is the terminal time and then my time domain becomes close interval to t and then this space Omega remains the same and then the sigma algebra on this product space we are going to consider this B 0 to t, product sigma algebra Ft, I mean the same thing actually, the notation is different but the same thing I want to mean, and here are BR. So, this is measurable for all t. So, if you put this condition then of course that if we ask that, not the joint thing but just for t fixed, this is omega the measurability is Xt Ft measurable? That is trivially true from here. So, here this is progressively measurable process.

Now, we would also revisit one notion of equal in law, when two processes would be called equal in law. So, earlier what you have seen that two processes are modification to each other or indistinguishability for the necessarily we want both the processes to be defined on the same probability space otherwise you cannot talk about it.

However, there are situations when two persons are coming from two different probability spaces, but still we can, we know that, they are similar, I mean they have similar distribution, they behave similarly. So, to talk about it what we do is that we consider, I mean we cannot consider an Omega here because they are coming from two different spaces possibly, but we can of course talk about the time points. So t1, t2…tn and at the same time points we observe X1, X2…Xn and Y1, Y2…Yn we can observe them, the value, the range value of the processes and then we asked if the probability that this the joint event Xti is in a Ai for i is equal to 1 to n.

So, here that means I am saying that Xt1, Xt2…Xtn this n-tuple, is in the Cartesian product of A1 cross A2 cross etc An and this probability and taking the value of that is matching with a probability that Yt1 belongs to A1, Yt2 belongs to A2 etc Ytn belongs An. That means the (())(27:01) Y1, Y2…Yn this interpel is in the Cartesian product, A1 cross A2 cross etc An.

If that is true, no matter what n you choose, finite n and no matter what time point you choose, all the time it happens then you say, X and Y have same finite dimensional distributions and then from the Kolmogorov's extension theorem, which I am not going to restate here, I assume that background, so implies that X and Y would have the same law. They have the same law so we would say that they have the distribution or same law.