Introduction To Rings And Fields
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Lecture – 34
Field extensions 2

Let us continue now. In the last video, we started talking about fields after defining field extensions which is the main object that we are going to study; we discussed when an element is algebraic or transcendental ok.

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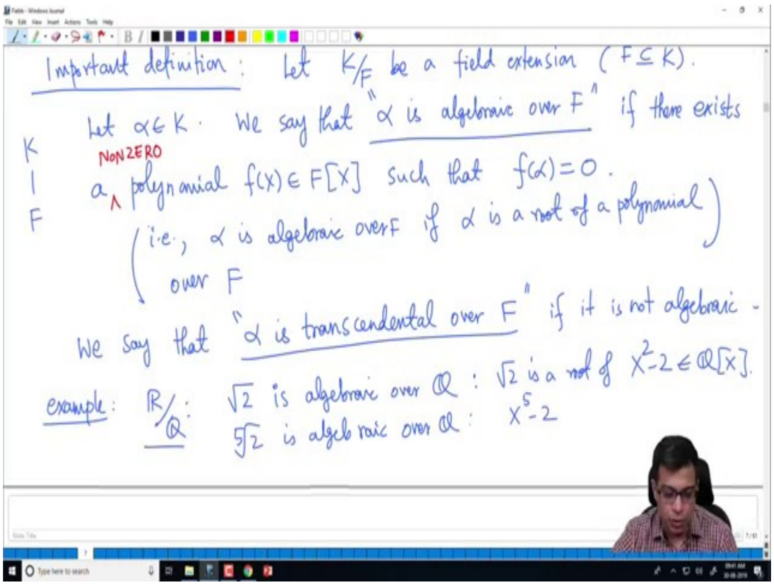

So, we say that an element of capital K which is the bigger field. So, K is the bigger field and F is the smaller field. We say it is algebraic if there is a polynomial below, which satisfies alpha. And if there is no such polynomial, we say that it is transcendental. Common examples are these we have root 2, fifth root of 2 i, then all algebraic over Q; pi and e are not algebraic over Q ok.

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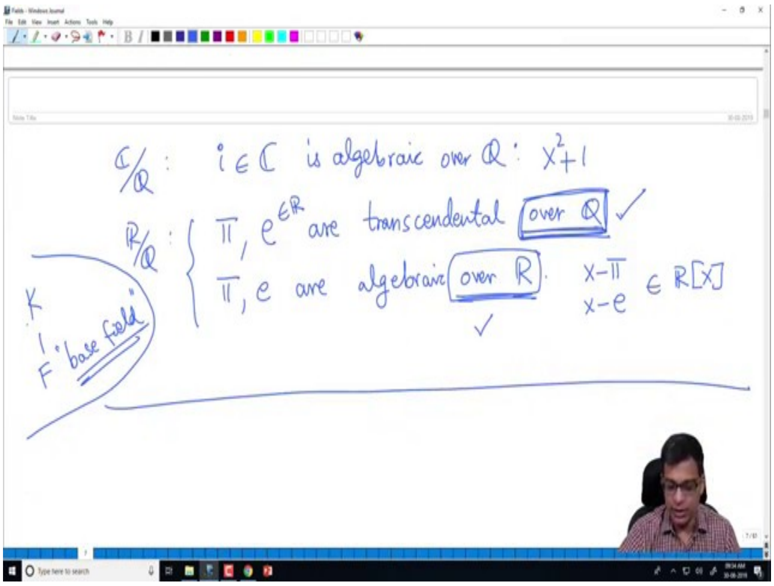

So, we are going to study now a more conceptual way of determining whether an element is algebraic or not.

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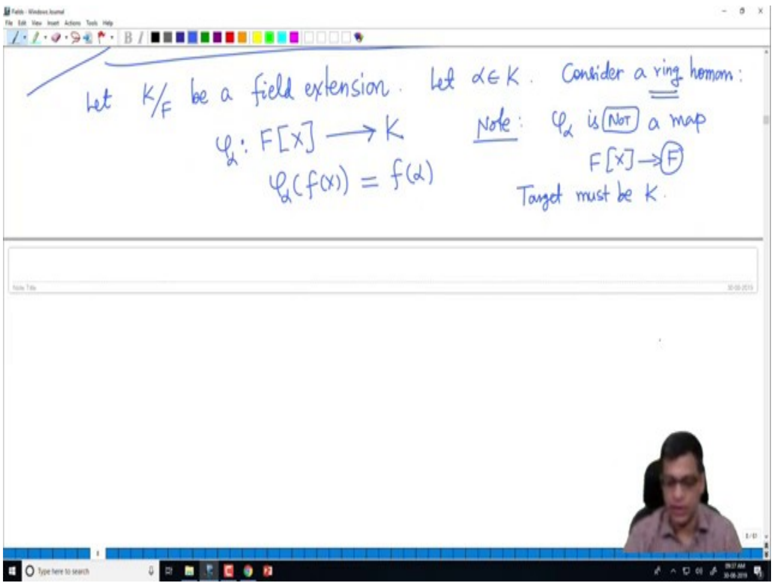

So, let us start again with a arbitrary field extension let K over F be a field extension ok. So, and let us choose alpha over K. So, this is going to be very typical to many things that we are going to do. We started the field extension and we will pick an element in the bigger field. So, what we want to do is consider a ring homomorphism which we discussed earlier. We consider ring homomorphism as follows. So, we consider the polynomial ring over capital F in one variable and K. So, this is a ring homomorphism, because F X is just a ring right, F square bracket X is a ring. K is a field, but F square (Refer Time: 01:55) is not a field in general. So, this is a ring homomorphism, and let us denote this by phi alpha because it depends on alpha.

So, I am going to use subscript alpha to denote that this is completely dependent on alpha. And what is phi alpha going to do it is going to take a polynomial and evaluating that polynomial at alpha. So, f x goes to f alpha. So, very important to note that phi alpha is not a map from F x to F right. So, it is not a map from f x x because F x to F, because when you compute f alpha though the coefficients of a f are in capital F alpha is not necessarily in capital F alpha is in K. So, the target must be K right. The target must be K because f alpha is in K, not necessarily in alpha, not necessarily in F.

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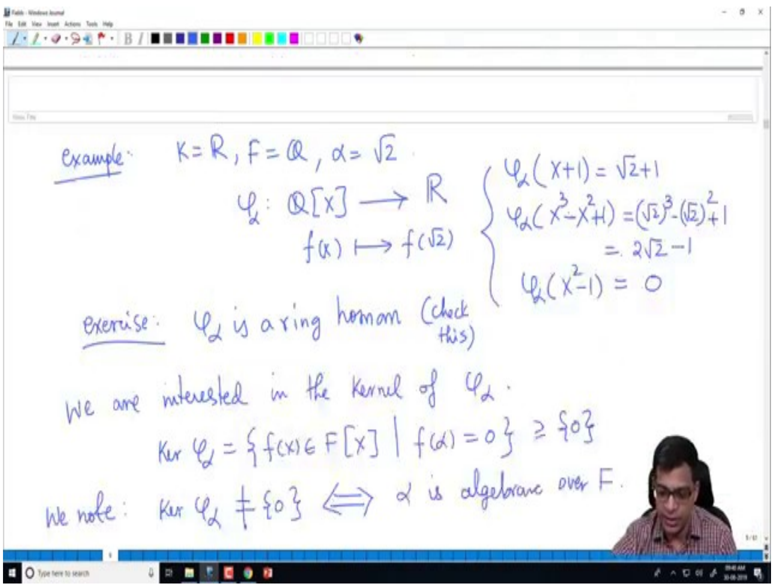

So, the example that we will consider is we can take K to be let us say R, and F to be Q and let us take alpha to be root 2 ok, so this simple example. So, what would be phi alpha, phi alpha would be a map from the polynomial ring over Q in one variable to R not to Q. And what will it do it takes a polynomial f x and evaluates it at root 2. For example, phi alpha of x plus 1 will go to root 2 plus 1; phi alpha of x cubed plus 1 will go to 2 root 2 let us say x cubed plus x cube minus x squared plus 1. It will go to root 2 cubed plus or root 2 cubed minus root 2 squared plus 1 which is 2 root 2 minus 2 plus 1, so that is minus 1, and where will x square minus 1 go, it will go to 0.

So, these are just some examples. So, I am going to take a polynomial with rational coefficients replace x by root 2. So, I will get an arbitrary real number. So, it is it is not a difficult exercise at all to show that phi alpha is a ring homomorphism. Because when you take two polynomials f and g, and take f plus g and evaluate at root 2, it will give you nothing other than f of root 2 plus g of root 2. Similarly, f g of root 2 is a f of root 2 times g of root 2.

So, I will leave this as an exercise for you check this, this is an easy calculation. So, phi alpha is a ring homomorphism. We are interested in the kernel of this all right. In this example for if you see x squared minus 1 is in the kernel what is the kernel of ring homomorphism things that map to 0. So, x squared minus 1 goes to 0, but x cube minus x squared plus 1 and x plus 1 are not in the kernel. What is the kernel? So, kernel of phi alpha is all polynomials. So, now, I am going to switch to the general notation. We have a capital F which is a subfield of capital K and alpha is in capital K, phi alpha is the map with sends f x to f alpha. What is the kernel, this is 0 f alpha is 0.

Now, as you can see this is intimately connected to the notion of algebraic or transcendental elements that are defined in the previous video. So, immediately we note kernel phi alpha is not just the zero element. Remember of course, it contains 0 kernel is an ideal in f x kernel is an ideal in f x by definition of kernel or not by definition, but it is an easy exercise that we did in the part when we discussed ring homomorphisms. Kernel F is an ideal, so it contains 0. It is non zero, it means it is strictly bigger f and only if alpha is a algebraic over capital F right. This is very easy because kernel f is nonzero means there is a known zero polynomial which has alpha is a root, so it is algebraic that is one direction.

If also alpha is algebraic there is a nonzero polynomial capital F such that capital F sorry nonzero polynomial small f whose root is alpha, so that means, it is in kernel phi alpha. So, now, I am just thinking that I may have omitted this when I talked about algebraic elements that I should have said here. We say that alpha is algebraic over F if there are exits a nonzero polynomial that must be mentioned right it is very important, because otherwise every element will be algebraic because you can always take this zero polynomial.

So, the whole trick is to demand a nonzero polynomial because pi of course, is a root of the zero polynomial over Q, but we still say pi is transcendental because there is no nonzero polynomial. So, if alpha is algebraic, then there is a nonzero polynomial for which alpha is a root that nonzero polynomial will be in the kernel of phi alpha. So, now this completely characterizes the two cases whether alpha is algebraic or not is determined by kernel of this map.

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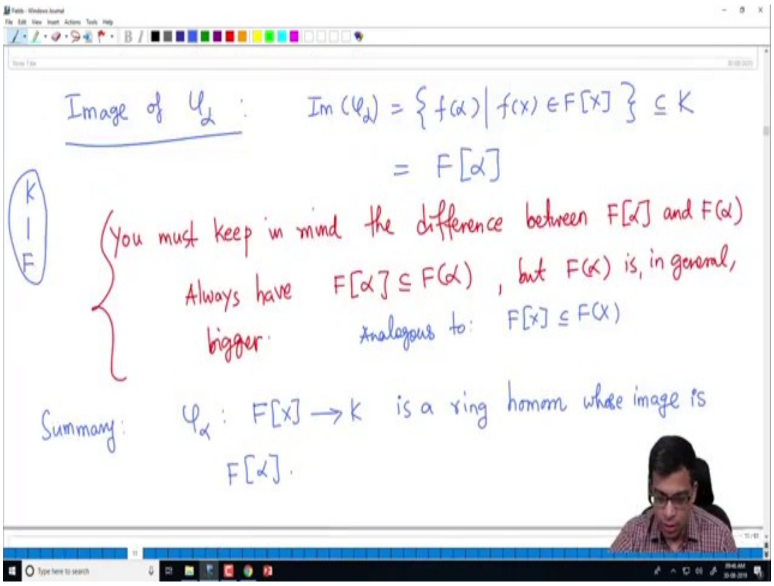

So, let us consider the two cases separately before that kernel is connected to the notion of algebraic or transcendental this way, what is the image of phi alpha. Let us explore this. What is the image of phi alpha? Image of phi alpha is all elements which are of this form, I am just formally writing this right, because phi alpha sends f x to f alpha. So, if you take f alpha and vary f x in capital F x you get image of alpha. And the notation is remember this looks very much like the field F round bracket alpha that we defined earlier F round bracket alpha is this ratio. Now, what we have is only the numerator denominator is not allowed. So, this is actually very similar to just polynomials in alpha. So, we are going to denote this by F square bracket alpha.

You must keep in mind the difference between F square bracket alpha and F round bracket alpha right. So, the difference is F square bracket alpha is obtained by taking polynomials over capital F and evaluating them at alpha taking all the elements. So, of course, this is in K. All this is with again I am in repeating this all this is with respect to a particular given field extension F square bracket alpha is all such things, F round bracket alpha is not just polynomials, but ratios of polynomials.

So, we always have F square bracket alpha contained in F round bracket alpha, but F round bracket alpha is in general bigger. So, this is the important remark that I want you to keep in mind, in general it is bigger. What makes it tricky is that many times it is actually equal that we will come to in a minute, but in general it is bigger.

So, this is exactly analogous to the construction of polynomial ring and its field of fractions which we denoted by F round bracket x. So, only difference is here X is a variable, here alpha is not a variable, but it is actually an element ok. So, now, let us comeback to this phi alpha is, image phi alpha is f alpha. So, what do we have now, so summary is that phi alpha from capital F x to K is a ring homeomorphism with kernel ok. So, with kernel we understand whose image is F square bracket alpha.

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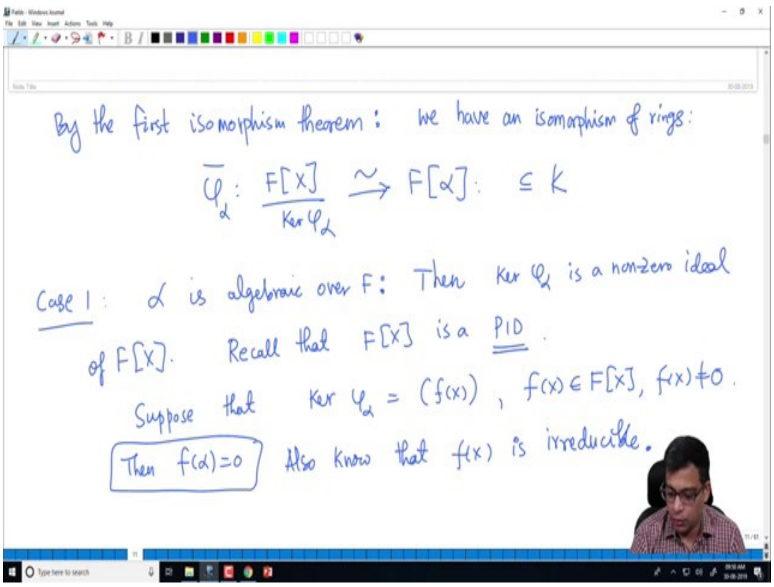

So, by the first isomorphism theorem, by the first isomorphism theorem, we have an isomorphism of rings which I will denote also by phi alpha for simplicity from F alpha F x modulo kernel phi alpha. So, let me maybe for simplicity clarity denote it this by f alpha phi alpha bar to F alpha right. This is exactly the isomorphism theorem which says that if you have a ring homomorphism the domain mod the kernel is isomorphic to the image. So, the F x mod kernel is isomorphic to the image which is F alpha ok.

Now, we are going to consider two possible cases. Case 1, alpha is algebraic over F. Then kernel phi alpha is a nonzero ideal of F x right by definition or rather by this observation that we made, whether alpha is algebraic or not is determined by this morphism homomorphism phi; alpha, kernel phi is nonzero if and only alpha is a algebraic over F. So, we are assuming that it is algebraic, so that is the case that we are considering kernel phi alpha is actually a nonzero ideal of F x.

Now, note that recall that f x is actually a PID. What is that; that means, every ideal in F x is a principal idea, because you can use Euclidean division and you with the degree as a size function and conclude that every ideal is generated by a single polynomial. So, suppose that kernel phi alpha is generated by f x, where f x is nonzero it is a polynomial in f x in nonzero polynomial right.

So, suppose this, of course we know that so then of course, f alpha is 0 all right, so that is because it is in the kernel. But every polynomial which has alpha is a root which has alpha as a root is actually a multiple of F that is what this sentence means kernel phi alpha consists of all polynomials which contain f as alpha as a root, and every polynomial kernel phi alpha is at multiple of f x. So, every polynomial which has which has alpha as a root is a multiple of f x.

Now, we also know that f must be irreducible, we know that it, it must be irreducible right because this is a consequence of how we prove that f x is a PID. We take all polynomials in that ideal look at the one with the least degree and we show that everything else is divisible by this. So, if this polynomial f x is not a reducible, it has a two factors g and h, but one of them must have alpha as a root because g, f alpha is 0, g alpha times h alpha is 0. But if f is equal to g h neither g, they cannot and if it is a proper factorization they cannot be written as a multiple of f. So, it must be reducible remember that. So, I am going to explain this. So, the reason is the following.

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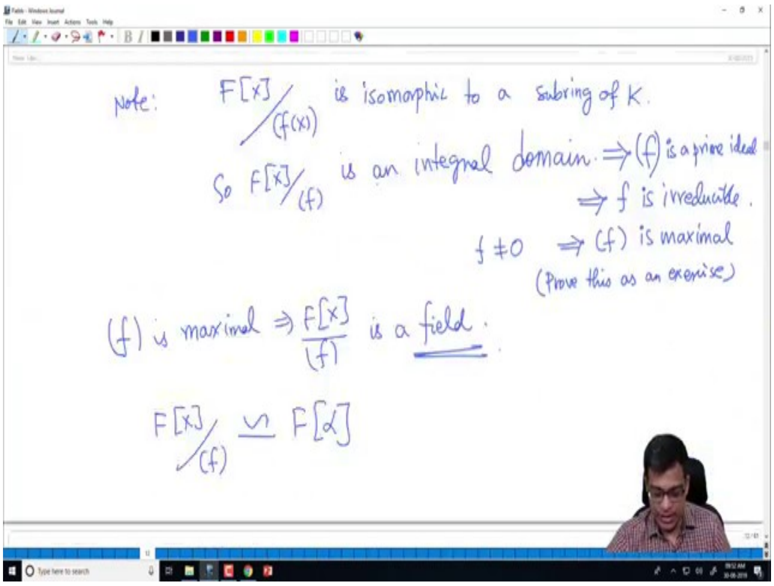

So, there are several ways of explaining this. So, note that we have F x modulo f x is isomorphic to a subring of right of course, it is because that is what we are saying here of course everything here is happening within K. So, f alpha modulo f x modulo kernel phi f which are now which I know now that it is generated by f x, f x F x modulo small f x is isomorphic to a subring of K. So, F x mod f x, I will just write it as f is an integral domain right.

A subring of an integral domain is an integral domain. Hence, f is a prime ideal, it is a prime ideal the ideal generated by f is a prime ideal, but this means f is irreducible. So, in a PID if you have a prime ideal, it is generated by a prime element which is actually an irreducible element. So, f is irreducible, so that is the explanation for my statement here, f x is irreducible. But if f is irreducible, ideal generated by f is actually maximal, of course, we are using here that f is nonzero. Remember that in a PID, if you have a nonzero element which is irreducible, it is ideal generated by that is maximal. This of course, I covered in earlier part, but you can prove this as an exercise is a good way to revise that material. So, it is a maximal ideal.

But once f is maximal, F x mod f is a field right, one equivalent characterization of a maximal ideal is that F x mod small f is in fact of field. So, now, let us take stock of where we are. We have F x modulo f is isomorphic to F alpha that comes from before first isomorphism theorem said F x mod kernel is F alpha isomorphic F alpha; kernel f alpha is small f. So, F x mod small f is isomorphic to F alpha, but this is already a field.

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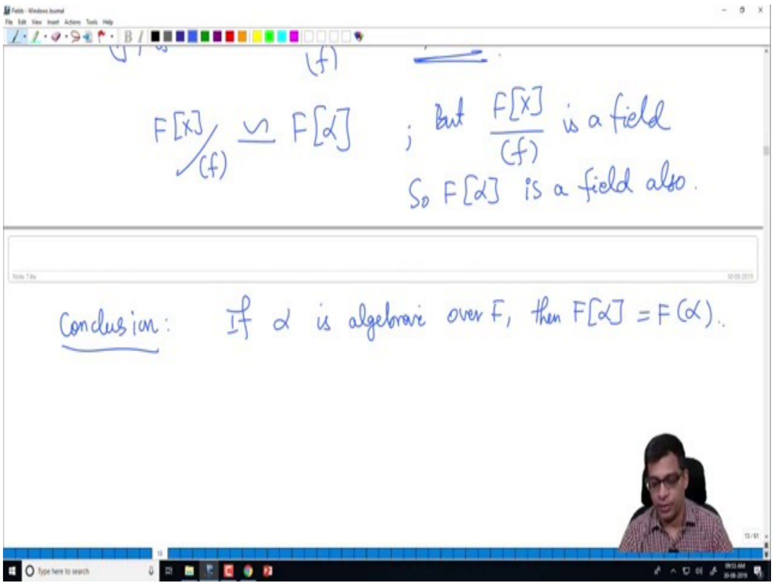

So, we have a field another ring which is isomorphic to that field has rings. So, F alpha is field right. If you have a two rings which are isomorphic, one is a field the other must be a field, because it is a ring isomorphism. If you take an what is the extra condition that one has to prove that F alpha is a field. All that is missing is inverses of elements. So, you take an nonzero element here. You take its image here that has an inverse, take that inverse and take its image, because it is an isomorphism the product here will be the product here images of the products, but one has to go to 1. So, everything here has a has an inverse. So, F alpha is a field.

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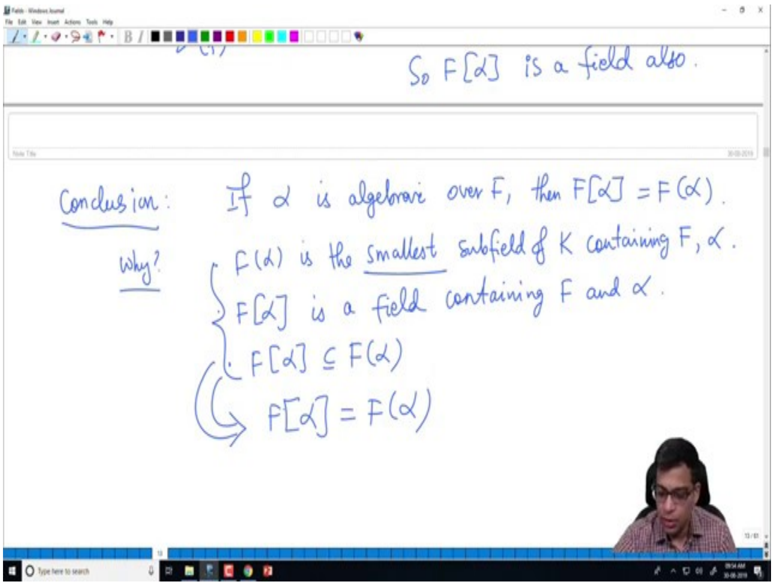

So, the conclusion of all this analysis is, if alpha is an algebraic element over F then F square bracket alpha is equal to F round bracket alpha. Remember earlier I emphasized the difference between F square bracket alpha and F round bracket alpha, in general I said they are different, but if alpha is algebraic they are same why are they same. Remember F alpha is the smallest field sub field of K containing F and alpha.

What smallest here means is actually that if any field contains capital F and alpha that will contain F alpha F round bracket alpha. But F alpha F square bracket alpha is a field containing of course it contains F and alpha right. And finally, F alpha is contained in F round bracket alpha F square bracket alpha is contained in F round bracket alpha, these three things together imply that F alpha is equal to F alpha F square bracket alpha is equal to F round bracket alpha. So, this is the weird thing in this. If alpha is algebraic, it suffices to consider polynomials in alpha and they automatically become they automatically form a field ok.

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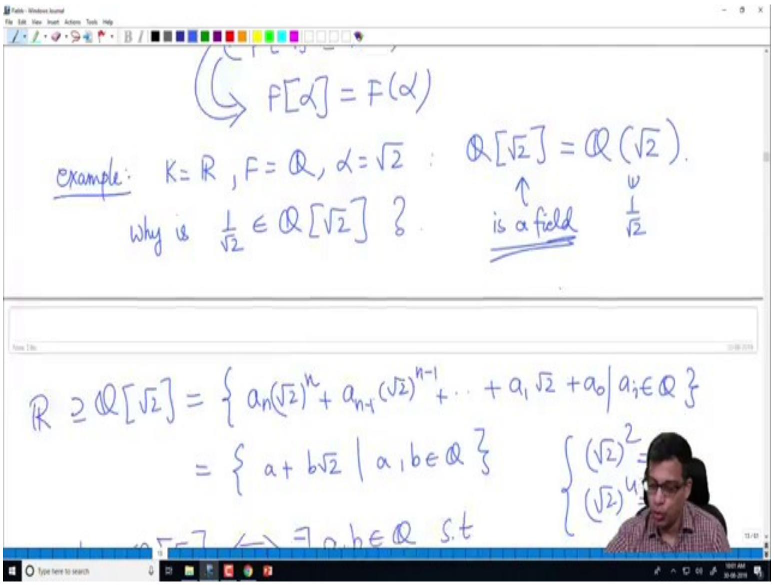

So, the reason is. So, take again K to be R, F to be Q. I am going to do this simple example often to illustrate the various concepts that we are doing. So, what we are now claiming is that Q square bracket root 2 is equal to Q round bracket root 2. Let us see what happens, why is this? On the face of it, Q round bracket root 2 clearly contains more elements than Q square bracket root 2 for example, 1 by root 2 is here. This only consists of things, which are polynomials in root 2, how can 1 by root 2 be there, but we are saying it is there. So, lets understand why is 1 by root 2 in Q square bracket root 2. So, why is 1 by root 2 in Q square bracket root 2?

So, what is Q square bracket root 2? So, it is a good time to recall what is square bracket root 2. So, these are polynomials like this, a 1 root 2 a 0, and what are ai's, ai's are in cube. And remember this is of course, sitting inside R. So, everything remember we have to keep emphasizing we have a bigger field under which all the stuff is happening, and we have a smaller field above which all the stuff is happening. So, here we are within R and above Q.

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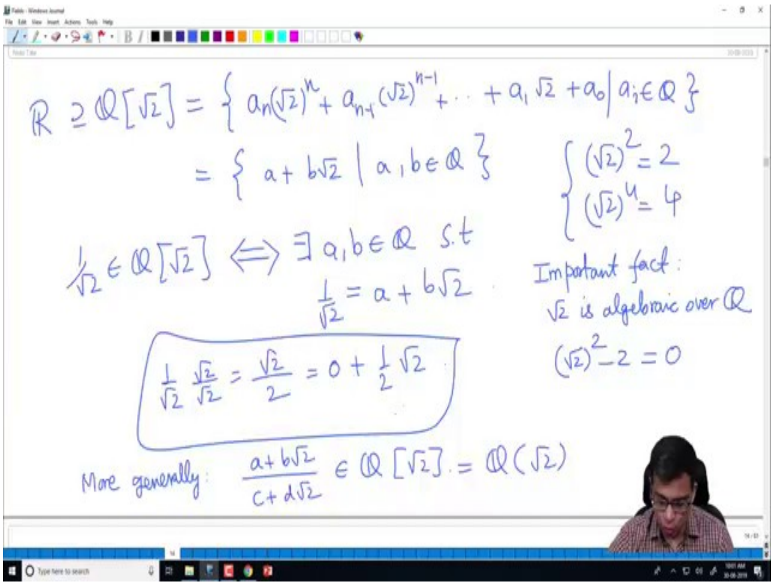

But of course, you can see that root 2 squared is going to be observed in a 0. So, I am going to really like this is a plus this is a math simpler description this, because root 2 squared is 2, root 2 4 is 4 and so on right. So, when you have other terms here, we can absorb them into the constant term. So, and we are only going to be left with a and b root 2. So, what we are saying is that 1 by root 2 is in Q square bracket root 2, but this means there exist a and b rational numbers such that 1 by root 2 is equal to a plus b root 2 right.

This seems impossible right, I mean how can it be, because here you are only taking a and b are rational numbers. So, the denominators there are integers, whereas here we have one by root 2 that has denominator root 2, how can we some we can get this. The point is we can get this and all we need to do is just play around with this a little bit. Ultimately the important fact is that makes this work is that important fact is root 2 is algebraic over Q, so that means, we know that root 2 squared minus 1 is 0 ok. So, root 2 squared minus 1 is 0.

So, let us use this as our tool to write this. So, how do you write this. So, for example, we know that sum. So, we want to use of under root 2 is algebraic right. So, root 2 squared is 1, root 2 squared minus 2 is 0. So, if you take 1 by root 2 and multiply both numerator and denominator by root 2. So, what we get is the whole point is root 2 squared becomes rational. So, when you do this, it becomes root 2 by 2 and that is it right that we can write as 0 plus 1 by 2 root 2 ok.

So, this is how we can write 1 by root 2 as rational number plus another rational number times root 2. So, 1 by root 2 is also in Q. So, more generally a plus b root 2 by c plus d root 2 as long as this is nonzero of course is in Q adjoined root 2. The point I want to emphasize that we do not need to prove this anymore because we have already shown that Q root 2 is a field Q root 2 is a field and c plus d root to is a nonzero element there. So, 1 by c plus d root 2 will be there.

So, a plus b root 2 is also there. So, any ratio is there. So, Q root 2 it is easy to describe Q root 2 the round bracket root 2 is actually just square bracket root 2. So, this is a very important observation. If you have an algebraic element, you have square bracket and round bracket are same in general as I wrote earlier in general of course, round bracket is bigger, but if the element is same, round bracket alpha is same as square bracket alpha. The situation is very different when you have algebraic elements. So, let us look a transcendental element. So, let us look case two.

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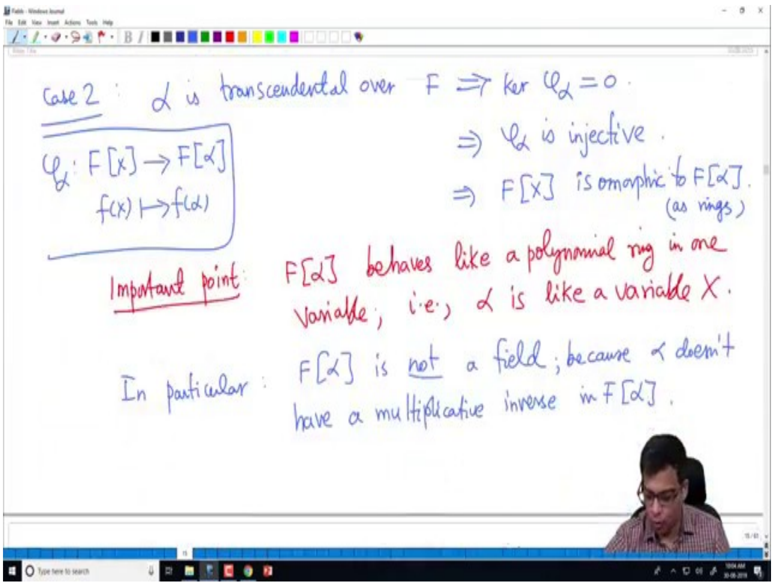

So, with this complete case one, so let us case 2. So, suppose alpha is transcendental over F, now this means kernel of phi alpha is 0. Remember phi alpha is the map from polynomial ring over F to K and in fact I am going to write this as F square bracket alpha because the image is something we have already analyzed. It is F squared bracket alpha, it sends a polynomial to F alpha. If it is transcendental, there is no polynomial which goes to 0, because there is no polynomial whose one of its root is alpha. So, kernel phi alpha is

0, that means, phi alpha is injective, so that means, F square bracket x is isomorphic to F square bracket alpha right. So, the important point here is point here f alpha behaves like a polynomial ring in one variable that is alpha is like a variable X right.

So, this is the important point because F x is isomorphic to F alpha as rings. So, I should try to as rings or another way of saying is that alpha is just like a symbol, X is a symbol alpha though it is actually an element it is like a symbol. So, in particular F alpha is not a field, F alpha is not a field right, because it is isomorphic to F square bracket x the F square bracket x is certainly not a field because x has no inverse. This is the crucial difference between algebraic and transcendental elements.

If root 2 is algebraic, root 2 has an inverse in the polynomial ring Q x square bracket root 2. But if alpha is not algebraic alpha does not have any inverse multiplicative inverse in F square bracket alpha. This is the crucial contrast between algebraic and transcendental elements if alpha is algebraic it has a root in it has an inverse in f alpha as we saw in the example of root 2 because root 2 does have an inverse in Q adjoints square bracket root 2. If alpha is transcendental, it does not have any inverse ok, so that is a fact because x goes to alpha under this map and xs now inverse. So, alpha cannot have any inverse.

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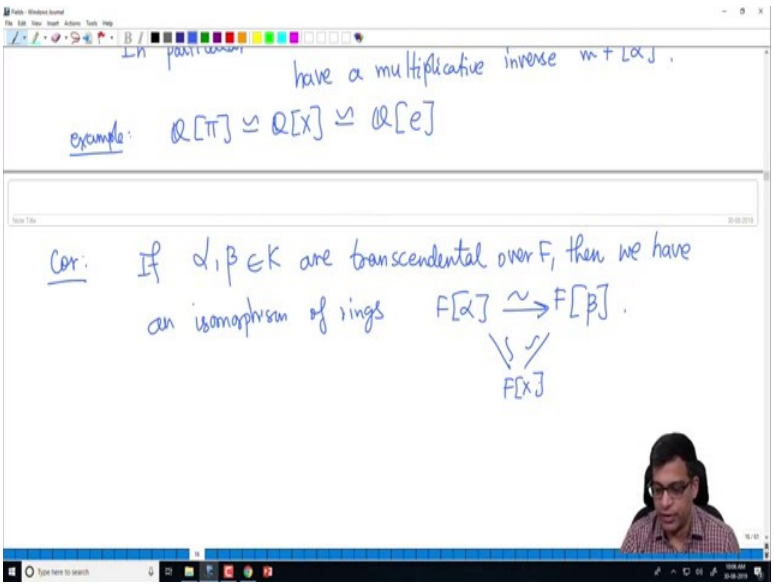

So, as an example we know that Q adjoined X is or Q adjoined pi is isomorphic to Q adjoined X because pi is transcendental I remarked earlier, but Q adjoined X is also isomorphic to Q adjoined e. So, here we have an important corollary is that if alpha and beta are transcendental over let us say there both in K or transcendental over F, then we have an isomorphism of rings F alpha F square bracket alpha to F square bracket beta right, because they are both individual isomorphic to F square bracket x right. So, they are mutually isomorphic now.

If alpha and beta are transcendental, algebraically there is no difference that is the conclusion, algebraically there is no difference. Of course, they could be a very different as real numbers pi and i, e are very different properties they have different definitions different properties. But algebraically there is no difference between any two transcendental elements.

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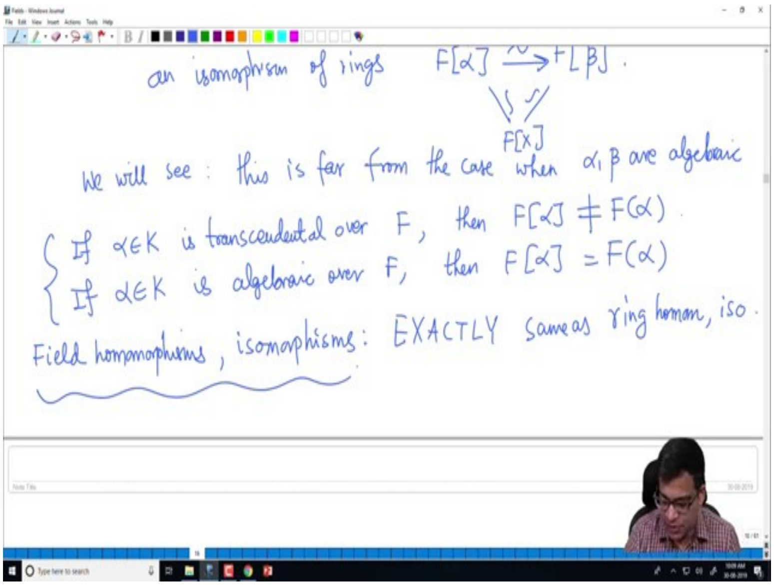

And we will see later in the next video this is far from the case when alpha, beta are algebraic. When they are algebraic, you will not get often that there are isomorphic ok. So, as we will explain in the next video. So, but in the case of transcendental elements, you have the isomorphism. So, also I should remark that as a consequence. If alpha is in K and it is transcendental over F, then F alpha is actually F square bracket alpha is actually not equal to F round bracket alpha. This is another deference between algebraic and transcendental elements F square bracket alpha is merely a ring and not a field ok.

So, it is F round bracket alpha is actually strictly bigger set than F square bracket alpha. If it is algebraic, I will just record this here just for summarizing this, so that you can keep this in mind. If it is algebraic over F then F square bracket alpha is equal to F round bracket alpha. So, this is a very different important distinction between algebraic and transcendental elements.

So, let me make a one final remark about field homomorphisms and field isomorphisms, because in the next video I am going to talk about these things field isomorphisms and homomorphisms. Field homomorphisms and field isomorphisms are exactly same as ring homomorphisms and isomorphisms ok. There is absolutely no difference.

So, what is a field iso, homomorphism, it is just a homomorphism between two fields. There is no additional condition, one has go to one and ok. So, I if you are two fields F and K, a homomorphism is basically just map of them as rings. It will automatically be a homomorphism of the multiplicative group also. Similarly, a field isomorphism is nothing, but a ring isomorphism it is a field homomorphism. So, it is ring homomorphism and its bijective.

So, these are really not new concepts. The additional property of a field that every nonzero element has an inverse does not change what homomorphisms are. It will turn out to be a ring homomorphism on a field will turn out to be a homomorphism of multiplicative groups also that, but that comes out of whatever actions we have for ring homomorphisms.

So, I am going to stop this video here. In the next video, we will continue the study of algebraic elements. And rest of the course is mainly going to be study of field extensions and algebraic elements in field extensions

Thank you.