Introduction To Rings And Fields
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Lecture - 21
Noetherian Rings 2

Let us continue now. In the last video I introduced the notion of Noetherian Rings.

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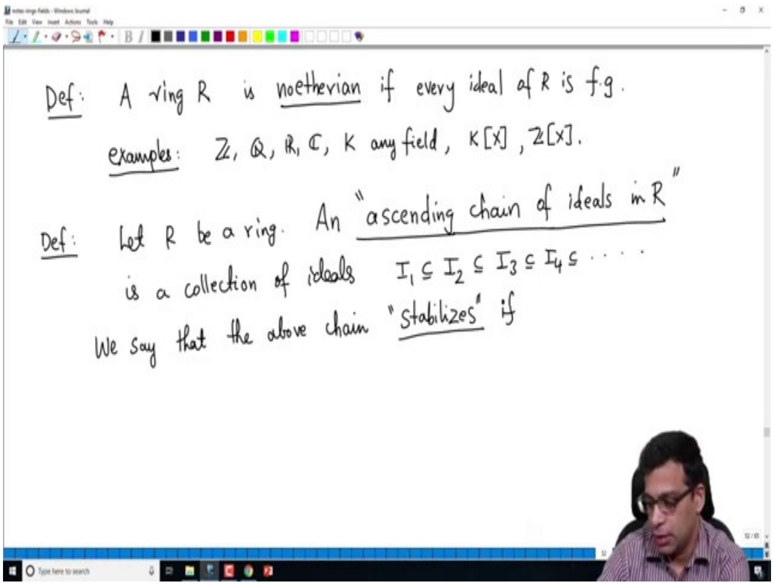

So, let me start by giving a quick recap of the definition. So, a ring is, a ring R always commutative ring with one is noetherian, if every ideal of R is finitely generated. And the important important examples are Z, Q, R, C, any field of course, any field is noetherian; any field, polynomial ring over in one variable or any field K is any field polynomial ring over Z.

This last example requires a little bit work, but that also is a that also is a noetherian ring and a non-example is a polynomial ring in infinitely many variables. So, in this video, I want to give some more notions more properties of noetherian rings. So, first of all I want to start with an equivalent characterization, so, another way to check if your ring is noetherian.

So, let us define the following notion, let R be a ring. An ascending chain of ideals in R, ascending chain of ideals in R is a chain is a collection of ideals like this, I 1 contained in I 2 contained in I 3 and so on. So, it keeps on going like this.

So, it is ascending because, I 1 is contained in I 2, I 2 is contained in I 3. So, I 2 is bigger, I 3 is bigger, I 4 is even bigger and I 5 is bigger than I 4 and so on. So, it is an ascending chain, it is a chain because each contains in it is each is contained in the next one so that is simply an ascending chain of ideals. We say that so, the definition continues, we say that the above chain stabilizes. We say it stabilizes if every sorry if there exists some integer n such that after n every ideal is same, such that I n is equal to I n plus 1 is equal to I n plus 2 and the rest. So, it is basically all ideals after the nth stage are equal.

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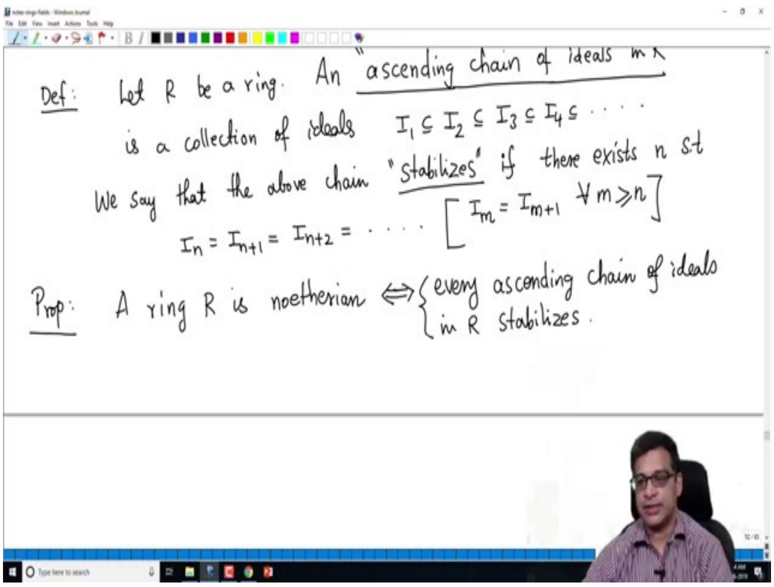

So, more precisely I should say that I m is equal to I m plus 1 for all m which are bigger than n. So, beyond the nth stage, all ideals are equal. So, though it is an infinite looking chain, it is only a finite chain. So, that is what we that is when we say it stabilizes. So, the proposition that I want to prove and this is useful because it will allow us to check noetherian rings in some cases is that, a ring is noetherian if and only if every ascending chain of ideals in R stabilizes ok. So, a ring is noetherian if and only if every ascending chain of ideals stabilizes.

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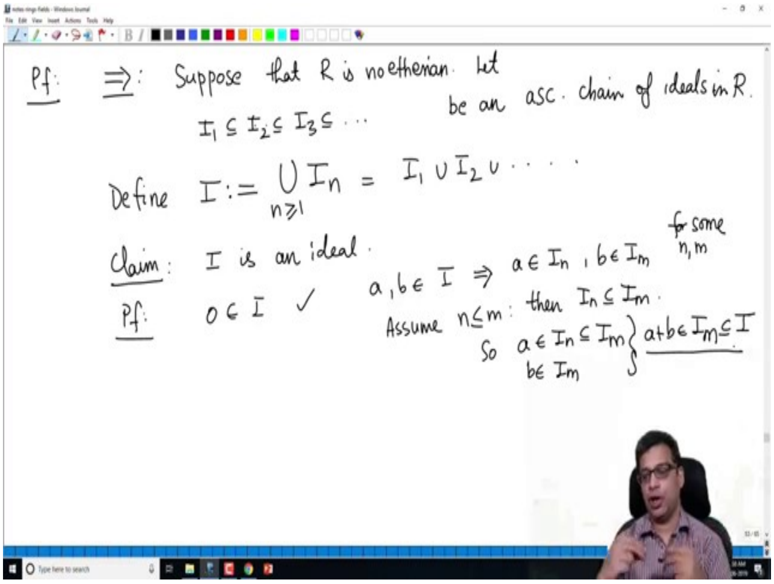

So, let us check this quickly, the proof is fairly straightforward. So, let us do this direction, suppose R is noetherian; so let us assume that R is noetherian and let us take a an ascending chain of ideals, let I 1, I 2, I 3 be an ascending chain.

So, let me write like this, ideals in R, let this be an ascending chain of ideals in R and our goal is to show that it stabilizes right that is the implication in this direction, I am assuming R is noetherian and I want to show that this ascending chain of ideals stabilizes. So, let us start with this ok. So now, what I will do is define I to be the union of I n ok.

So, what I am doing is, I am taking the union of all of these, just a set theoretic union to begin with. So, claim is that this is an ideal. So, in general the union of ideals is not an ideal because in the ring of integers, the ideal generated by 2 union the ideal generated by 3 is not an ideal because 5, 2 and 3 are in there in the union, but 2 plus 3, 5 is not in the union because 5 is not in 2 and 5 is not in 3. But if you take a chain of ideals where union is an ideal.

So, let me quickly prove this, what do we have to show. For example, we have to show that 0 is there, but that is OK, because 0 is in I n of course, for every n so 0 is there. Let us say a, b are in I, the main things to check care if two elements are in the ideal, their sum is in the ideal so, sum is in the set. So, let a comma b be two elements in i; that means, a belongs to I n and b belongs to I m for some n and m right.

I is defined to be the union, you take two elements so; that means, if something is in the union; that means, it is in some of the I ns. So, maybe a is in I n and b is in I m, but we know that so, assume I should write. So, assume without loss of generality, n is less than equal to m of course, m could be less than equal to m, in which case the argument goes in a similar way.

So, I n then is contained in I m right because I 1 is contained in I 2 is contained in I 3. So, if n is less than m, I n is continued in I m. So, a belonging to I n is actually in I m and b also is contained in I m of course, by definition. So, a plus b is in I m, but I m is contained in I because I m is one of the factors whose union is I. So, a plus b is in I, this is where the chain is important. One element is in I n, one element is in I m unless you are able to compare I and I n and I m, you cannot conclude this. So, a plus b is in I m.

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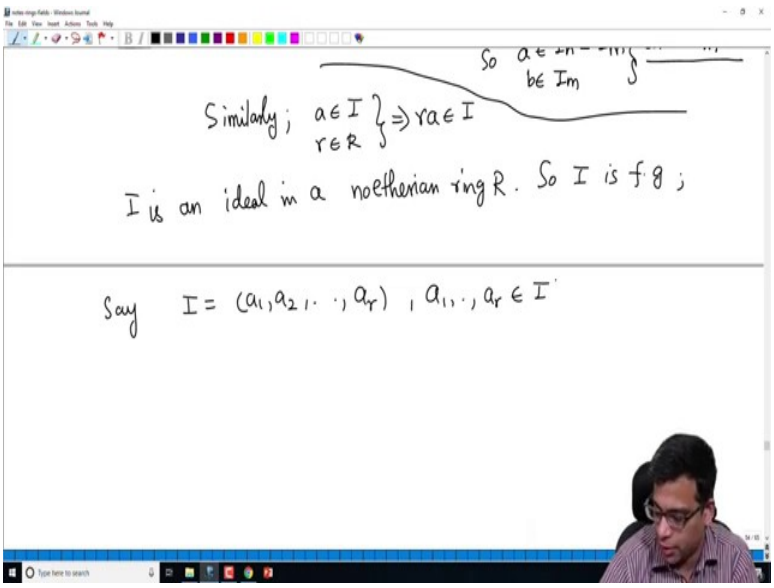

Similarly, so, I will leave this for you, you can quickly check this if needed stop the video and check this, if I is in I, a is in I, r is in R this implies, ra is in I. In fact, this is trivial; this does not require a chain property.

So, this proves that I is an ideal. So, this we have proved, but I is an ideal in a noetherian ring right, R is noetherian that is our assumption. So, I is finitely generated right by definition remember what is the definition of noetherian ring, every ideal is finitely generated. So, say I is equal to a 1, a 2,.., a n; let me call it a r where a 1 to a r of course, are elements in the ideal.

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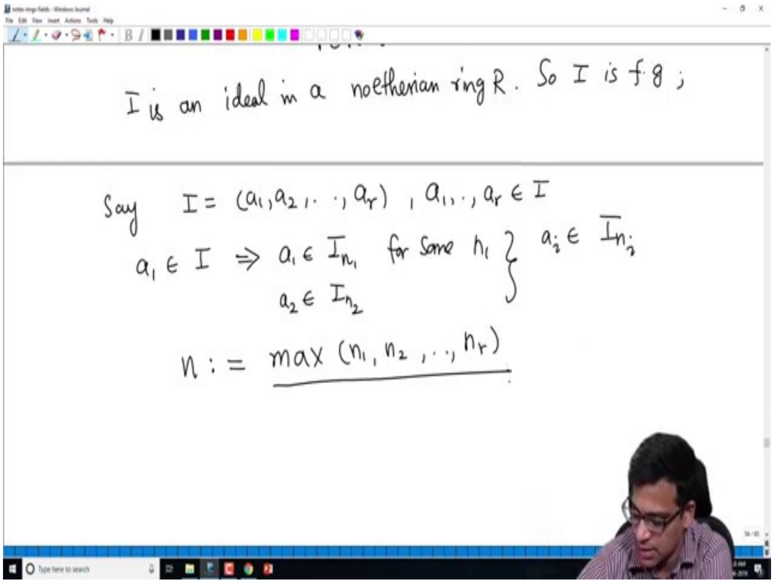

So, let us say I a 1 through a r generate I, but now a 1 belongs to some I, a 1 belongs to I means a 1 is in I, let us say n 1 for some n 1 because I is a union of these I 1, I 2, I 3 and so on. So, every element of I is in one of these let us say I n 1. Similarly, a 2 is in I n 2 and so on. So, in general a I is in I n I, but now I want to take n to be the max of n 1, n 2, n r.

So, a 1 is an I n 1, a 2 is, n 1 could be 100, a 1 could be a in I 100, a 2 could be in I 23, a 3 could be in I 42, a 4 could be I 205 and so on.

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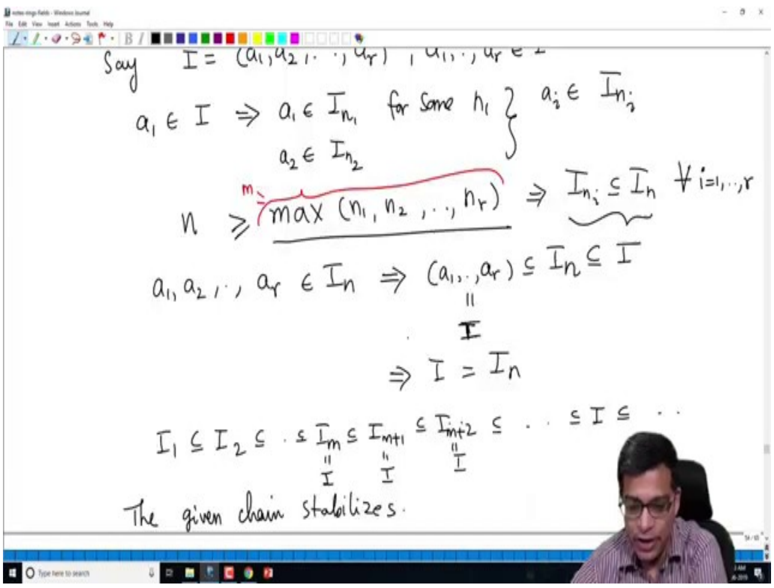

So, I am taking the maximum of all the indices; that means, a 1, a 2, a n, remember this means that I n I is contained in I n for all I from 1 to r because n is bigger than n 1, n is at least as big as n 1 so, I n 1 is contained in I n. Similarly, n 2 is at least n 2 is less than equal to n so, I n 2 is contained in I n. So, each of them is contained in I n; that means, a 1, a 2, a n are all in I n because they are all in I n I which is further contained in I n. This means the ideals generated by a 1 through a n is contained in I n right because if an ideal contains a bunch of elements, every linear combination of those elements is by definition in that ideal.

So, the ideal generated by a 1 through a n is contained in I n, but this is equal to I, by hypothesis I is equal to a 1 through a r. I should really write r here. So, this is equal to this, but I n is also contained in I right because I n is one of the factors whose union is I. So, I is contained I n, I n is contained in I; that means, I is equal to I n.

In fact, I should not define n like this if a n is at least this number, max of this then this argument holds so; that means, if you look at this chain I, I 2 and this number let us say is m. So, m is that number, you have I m then I m plus 1 and this is the original chain, but this we have just proved is I, this we have proved is I, I m plus 2 is I, at every stage we have I. This is exactly the meaning of a chain stabilizing.

So, what we have done is you are taking a chain; we took their union which is an ideal we argued. Once it is an ideal, we have shown that we do not need to show it is an assumption, remember R is noetherian. Once it is an ideal, it is generated by finitely many elements, the key word here is finitely many elements. Because there are finitely many elements and each of them belongs to one of the ideals I n, we can eventually reach a point where all the generators are in one particular I n and then after that, we get nothing new you get just the ideal I.

So, the given chain stabilizes. So, this completes the proof of one direction right, we wanted to prove that if a ring is noetherian, every ascending chain of ideals in R is going to stabilize.

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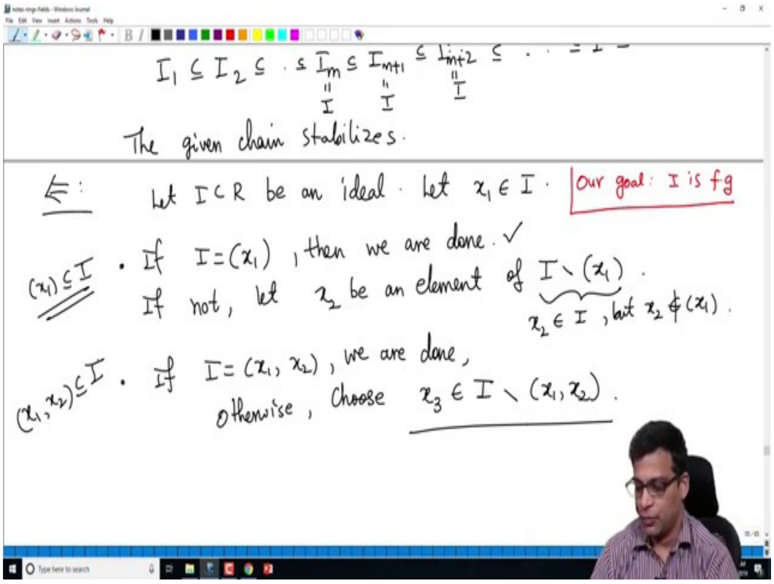

So, now let us go the other direction. We assume that every ascending chain of ideals stabilizes and we want to show that every ideal is finitely generated, OK this is easier. So, let us take an ideal and let us choose an element of I. So, let x 1 be any element of I. Of course, there are elements of I, I is a non-empty set right, an ideal is by definition going to contain the 0 element so it is non-empty. So, I can take an element.

If I is x the ideal generate by x 1 then we are done, why is that? If I is generated by single element it is of course, finitely generated. So, remember our goal is to show I is finitely generated, our goal is to show that I is finitely generated. You take an element if it just happens that I is generated by that, we are done because it is finitely generated by that single element. If not, let x 2 be an element of I minus x 1 remember x 1 is in I.

So, the ideal generated by x 1 is certainly going to be contained in I; if they are equal, we are done; if not, I is strictly bigger than the ideal generated by x 1. So, we take an element that is in I not in x 1 so; that means, x 2 is in I, but x 2 is not in x 1 ok. Now, we continue the induction step here if I is x 1, x 2 we are done.

Otherwise see if I is x 1, x 2, we are done because I is generated by two elements so, I is finitely generated which is our goal. Otherwise choose x 3 inside I minus (x1, x 2) right and in the second stage, the ideal generated by x 1 x 2 is contained in I because x 1 x 2 are elements of I. So, the ideal generated by x 1 x 2 is contained in I. If it is equal to I, we are done we stop the process. If not, I is strictly bigger than the ideal generated by x 1 x 2.

So, we can we can choose an element in I that is not in x 1 minus x 2 and we keep doing this.

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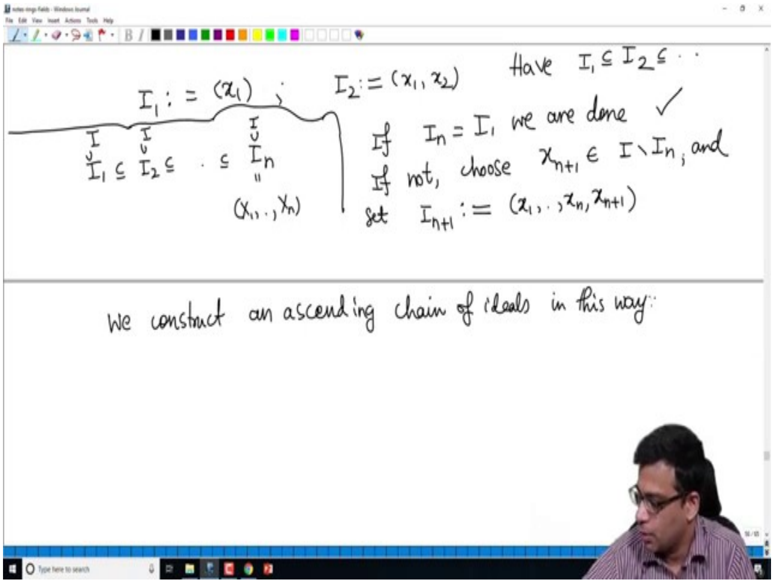

So, I am going to define I 1 to be the ideal generated by x 1, I am going to define I 2 to be the ideal generated by x 1 x 2. So, remember I 1 is contained in we have of course, I 1 is contained in I 2 because I 1 is just generated by x 1, I 2 is generated by x 1 x 2.

So, every element in I 1 which is a multiple of x 1 is already in I 2 and now you can see that we can keep doing this. So, let us say we have constructed I 1, I 2 up to I r or I n, how do we construct I n, it is generated by x 1 through x n and suppose if I n is equal to I, we are done because again its finitely generated by n elements. If not, choose x n plus 1 inside I minus in I n, I n is certainly contained in remember each of these is contained in I.

Every ideal that we are constructing is a sub-ideal of I, if it is equal to I, we are done, if not, we can choose an element which is in I, but not in I n plus 1 and define now having chosen I n x n plus 1, we define and we set I n plus 1 to be x 1 through x n comma x n plus 1. So, we can continue this process so and we construct and we construct an ascending chain, ascending chain of ideals in this way.

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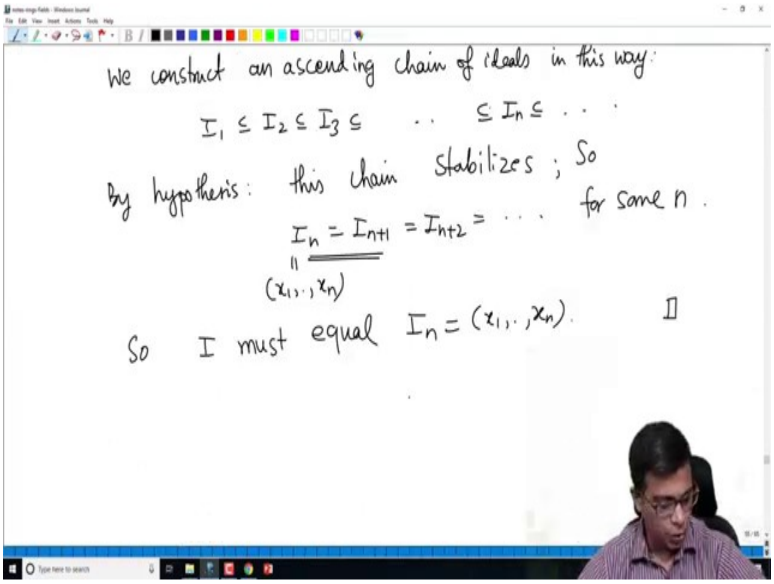

So, what we have is I 1 contained in I 2 contained in I 3 contained in I n contained in I n plus 1 and so on. And if you now use the assumption, by hypothesis this chain stabilizes right that is the hypothesis. So, I n equals to I n plus 1 equals to I n plus 2 for some n. So, there is some stage in this chain that you do not get a you stop getting new ideals, you keep getting the same ideal every time, but; that means, remember each ideal that we are considering in this case is a finitely generated ideal.

So, I must equal I n which is, why not, why must it equal. I will just say this in words, I must equal in because if it is not equal to I n, there is some element of I that is not in I n and we use that to define I n plus 1. So, I n plus 1 would be in fact, strictly bigger than I n and it will not be equal to I n which contradicts this assumption so that is the completion that completes the proof.

So, I must equal I n at some finite stage so; that means, I is finitely generated. So, a ring is noetherian if and only if, every ascending chain of ideal; every ascending chain of ideals stabilizes this is useful. So, I will now give two applications of this you have two statements about one example and one statement about noetherian rings using this alternate characterization of noetherian rings.

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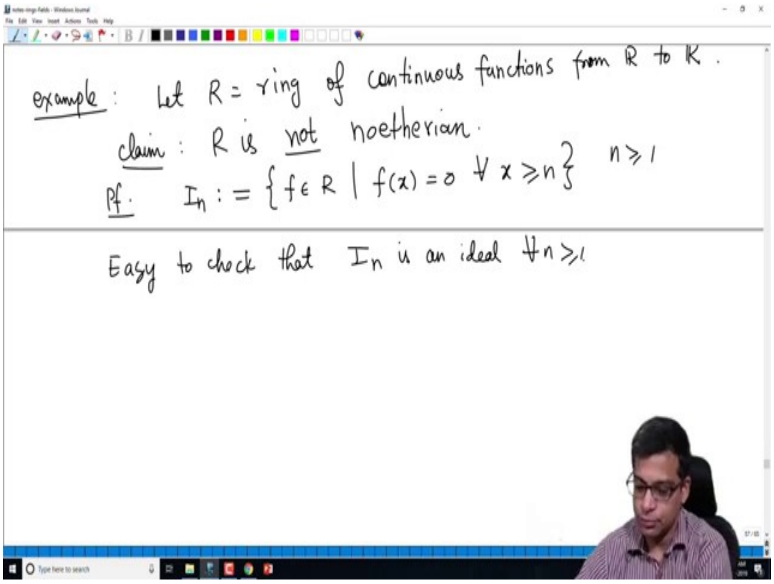

So, one example I want to give is the following. So, let us define R to be the ring of continuous functions. So, let me write it, continuous functions from the real numbers to real numbers and if you remember from one of the earlier earliest videos in the course, we gave this as an example of rings because you can add two continuous functions because the target is R, we can add f plus g to be f plus g of x to be fx plus gx.

Similarly, we can multiply two continuous functions and it turns out to be a ring. We claim that R is not noetherian, R is not noetherian. So, this is another example of something that is not noetherian, we in the last video looked at the polynomial ring in infinitely many variables that is one example, this is another example. So, why is this true?

So, to prove non-noetherianess, we either have to produce an infinitely generated ideal, more precisely we have to produce an ideal which is not finitely generated or we have to produce an ascending chain of ideals that does not stabilize so, we are going to do that. So, we are going to construct an ascending chain that is not going to stabilize. Let us define I n to be all continuous functions from R to R such that, fx is 0 for all x at least n ok. So, these are continuous functions which vanish beyond the point n.

So, it is an easy exercise to check that I n, so I am defining this for every positive integer, check that in is an ideal for all n, that is because if you add two functions which have this property; that means, their identical is 0 to the right of n; their sum also has the property.

If you take a function which is identically 0 to the right of n and multiply by any function, it will continue to be 0 to the right of n.

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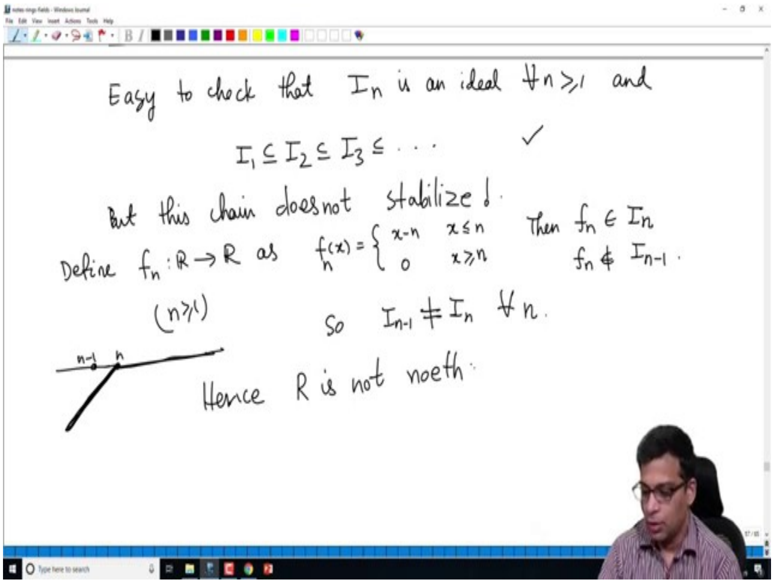

So, it is easy to check that it is an ideal and also easy to check that you have an ascending chain like this right because I 1 consist of functions which are 0 to the right of 1, I 2 consists of functions which are 0 to the right of 2. So, if something is 0 to the right of 1, it is also 0 to the right of 2 so, suddenly we have this change.

So, we have an ascending chain. So, we want to show that it does not stabilize, but this chain does not stabilize. If it does not stabilize, R cannot be noetherian because if R is noetherian, every ascending chain of ideals must stabilize. So, why does it not stabilize,? I claim that each stage we get a strictly bigger ideal. So, define a function so, I am going to first draw a picture.

So, define f 1 as follows f n. So, we can simply take any function you want up to n, this is a graph of that f so I am drawing the graph first because it is easy to see. So, up to n anything that is non-zero and to the right of n, it is 0. For example, this is just one example so, we can define x to be fx to be fnx to be let us say x minus n where x is less than equal to n, 0 when x is greater than equal to n that is this function right.

So, this function then f n is of course, continuous and it is in I n because it is 0 to the right of n, but f n remember is not in I n minus 1 because if you take n minus 1, it is not 0 on that x-axis right, it is not 0 between n minus 1 and n so; that means, that. there is a function which is I n, but not in I n minus 1. So, I n minus 1 is not equal to I n for all n, we can define this function for every integer at least one.

So, I n minus 1 is not equal to I n, I n minus 1 is strictly contained in I n. So, you have a non-stabilizing ascending chain of ideals. Hence, by the previous exercise previous proposition, R is not noetherian. So, in some sense, this example as well as the previous example are not noetherian because, they are too big in some sense. They are very big; there are ideals that are very big in the first case and there are there is an ascending chain of ideals which does not stabilize.

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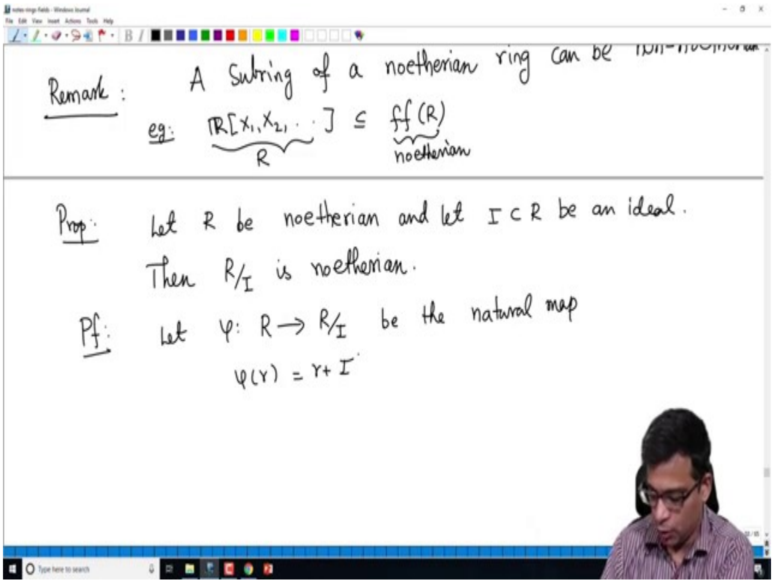

But interestingly now, there is the following maybe I should not write it as a remark, I should not write it as an example, but remark a subring of a noetherian ring can be non-noetherian. What I am saying is that in general, subring of a noetherian need not be noetherian as example would be remember the polynomial ring in infinitely many variables that we looked at, that we looked at in the previous video and that is not that is not noetherian because it has an infinitely it has an ideal which is not finitely generated.

But if you look at the polynomial this is certainly a integral domain, polynomial ring R is an integral domain. So, we can look at the field of fractions of R. This of course, is a noetherian ring because it is a field; any field is noetherian because it has only two ideals. So, both are finitely generated.

So, if any field is noetherian; however, this ring which is a subring of this is not noetherian. So, not non-noetherian ring are strange. So, you can have a noetherian ring containing a non-noetherian ring, so that can happen as this example shows; however, we do have a nice result nice proposition we can say, let R be noetherian and let I be an ideal of R, then the coefficient ring is noetherian. So, I am not going to give the full proof, it is a good exercise for you to do.

It just uses so, let the let phi be the natural map from R to R mod I, what is the natural map remember, it sends an element r to r plus I, the co-set corresponding to r. So, this is a natural ring homomorphism, so it is an onto ring homomorphism.

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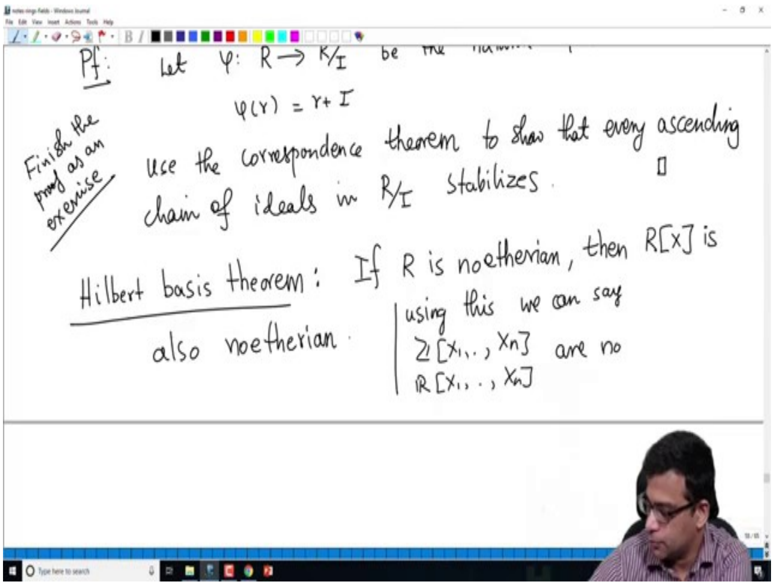

Use the correspondence theorem ,that is all I will say, use the correspondence theorem to show that every ascending chain, every ascending chain of ideals in R mod I stabilizes. Remember, a ring is noetherian, if and only if every ascending chain of ideals stabilizes in that ring. So, to show that R mod I is noetherian, it is enough to show that every ascending chain of ideals in R mod I stabilizes. So, now, how do we use the correspondence theorem to show, that take a chain in R mod I, take the inverse image basically, just take the inverse images of each ideal in that chain.

Now, that gives a chain of ideals in R. So, this is because, the correspondence theorem says that the bijection between ideals of R mod I and ideals of R that contain I is an inclusion preserving bijection. So, if you have J 1 containing J 2 containing J n which is a chain in R mod I, phi inverse J 1 will be contained in phi inverse J 2 which will be contained in phi inverse J 3 and so on. Now, that is an ascending chain in R, but R is noetherian so, it stabilizes; that means, phi inverse J n is equal to phi inverse J n plus 1, but that means, now again using the correspondence theorem, the ideals J n must equal the ideal J n plus 1 because if they were different the phi inverse would also be different ok.

So, I this is a hint for you, but finish the proof as an exercise, it is a good exercise to make use of several facts that we have so far learned in this course. The correspondence theorem, the equivalent definition of noetherian rings and so on. So, this proposition is nice because it gives us a way to construct new noetherian rings if you have a noetherian ring.

If R is a noetherian ring, we can construct a new noetherian ring by just taking the coefficient. That still leaves the question of how to construct noetherian rings to begin with. What do we know, so far we know Z is in noetherian, any field is noetherian, polynomial ring over a field is noetherian, but the most important way for us to construct new noetherian rings is what is called Hilbert basis theorem which I will prove in the next video. So, let me just end this video by stating this as a preview of what will come next.

So, this says that if R is noetherian then the polynomial ring in one variable in over R is also noetherian. So, this in particular we will prove that Z x is noetherian though you could do that directly just using the definition, but now and also it covers the case of a field adjoined x, K is a field then K polynomial ring over K in one variable is noetherian that is old example that is not new, but it Hilbert basis theorem does giving you examples.

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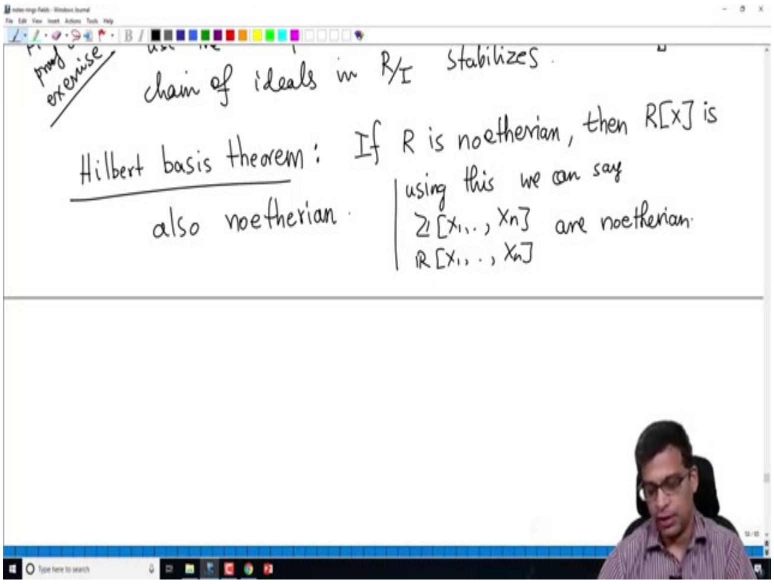

This will now in particular tell me that, we can say Z X 1 ... X n or R X 1 ... X n are noetherian. We cannot put infinitely many variables of course, then it will not be noetherian as we know, but finitely many variables it is noetherian and for this we need Hilbert basis theorem and how do we get this, this is simply because polynomial ring in n variables is simply a polynomial ring in one variable over the polynomial ring in n minus 1 variables. So, I am going to do this in more detail later so, do not worry about it, but once you can adjoin one variable to preserve noetherianess, you can keep doing this and any finitely many variables you will preserve noetherianess.

So, let me stop the video here, we have in this video looked at various examples of noetherian rings, we looked at an alternate characterization of noetherian rings using ascending chains of ideals and we showed that if a ring is noetherian, any quotient ring R mod I is also noetherian. In the next video, we are going to prove the Hilbert basis theorem.

Thank you.