Introduction To Rings And Fields
Prof. Krishna Hanumanthu
Department of Mathematics
Chennai Mathematical Institute

Lecture - 20
Field of fractions, Noetherian rings 1

Let us continue. In the last few videos we looked at a prime ideals and maximal ideals and an important class of rings called integral domains.

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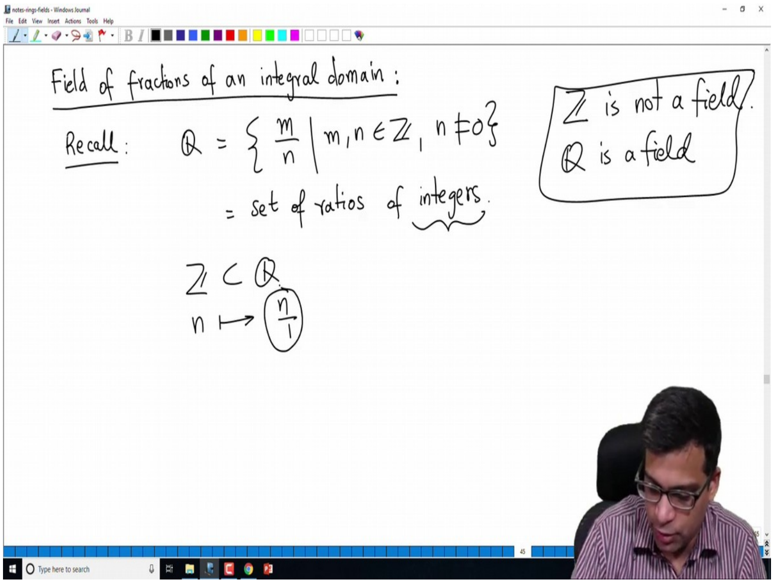

So, today's video I will start with a quick introduction to field of fractions of an integral domain. So, this is a very important concept, it is very easy to define this and the primary focus of this video is actually going to be noetherian rings that I will define soon, but I want to quickly introduce the notion of field of fractions of an integral domain.

So, let, so before I define this in general for an integral domain, let us recall the construction of rational numbers from the ring of integers. So, Z remember Q is all elements of this form where m and n are in Z and n is not 0. So, I want to express Q as somehow coming from Z; so Q is really set of ratios of integers. So, you take integers and you take their ratios and all possible ratios so you get Q. And what is happening in this process is so what you started with, Z is not a ring not a field right, we started with something that is not a field. But what we obtain is a field; that is a crucial point that I want to emphasize.

Z is not a field because it does not contain all ratios right in some sense because for example, 2 which is an integer does not have a multiplicative inverse; if it was there it would be 1 by 2. So, how do you introduce an inverse for 2? You simply add 1 by 2 to your set. So, similarly you add 2 by 3, you add 100 by 141.

For example, so you can add all ratios and Z is of course, sitting inside Q, as n is essentially thought of as n by 1. So, you have an integral domain, which is not a field and you have introduced all the ratios into that set and made it a field.

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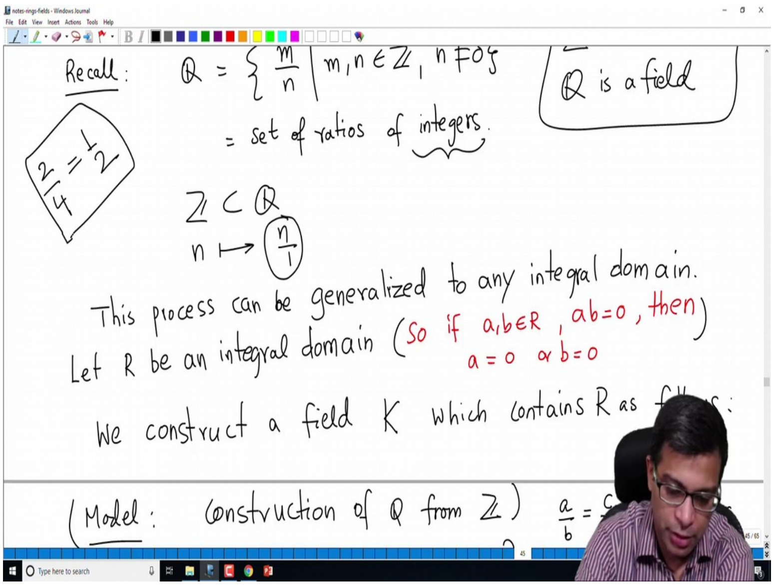

So, now this process can be generalized, to any integral domain so, this is what I want to explain now and the process is exactly the same. So, let R be an integral domain, let us recall quickly what is an integral domain? An integral domain is, so in other words, so if a comma b are in R and ab is 0 then, a is 0 or b is 0 right. So, every time you have a pair of elements from the ring whose product is 0, then one of them must be 0, this is an integral domain. We construct or we define a field let us denote it that by K which contains R as follows. So, I am going to define a field which is going to be a bigger ring, in other words it contains R as follows.

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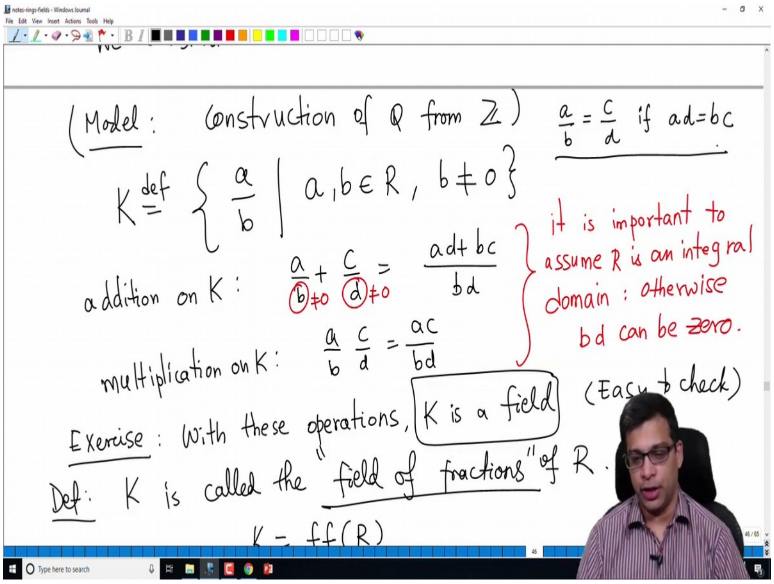

So, the model is construction of Z, Q from R from Z. So, that is the model you have to keep in mind; exactly the same model I am following for all integral domains. So, how do we define K? This is simply exactly as in the case of rational numbers from integers; we take all ratios. And of course, we do not want b to be 0, we do not want to write something by 0.

So, we take all possible ratios of elements of R and call that as set K, so K is by definition this. So, now, I want to make K a field; first of all, we have to define two operations on it, addition and it is exactly what you would expect that you are familiar with rational numbers. So, if you gave if you have two ratios how do you add them? You simply add them like this and what is multiplication, this is simply a by b times c by d is ac by bd.

And here is where you see that it is important to assume R is an integral domain right, you see that because if it is not; otherwise bd can be zero and then we will have a problem. We have b and d are non 0 right these are non 0 this is not equal to 0 this is not equal to 0. Because we are considering two elements of K, so a by b and c by d are elements of K; that means, b and c d are not 0.

In order to define the sum and product we will need this also to be non 0 and this is what integral domain guarantees us. So, we assumed that we are working with an integral domain.

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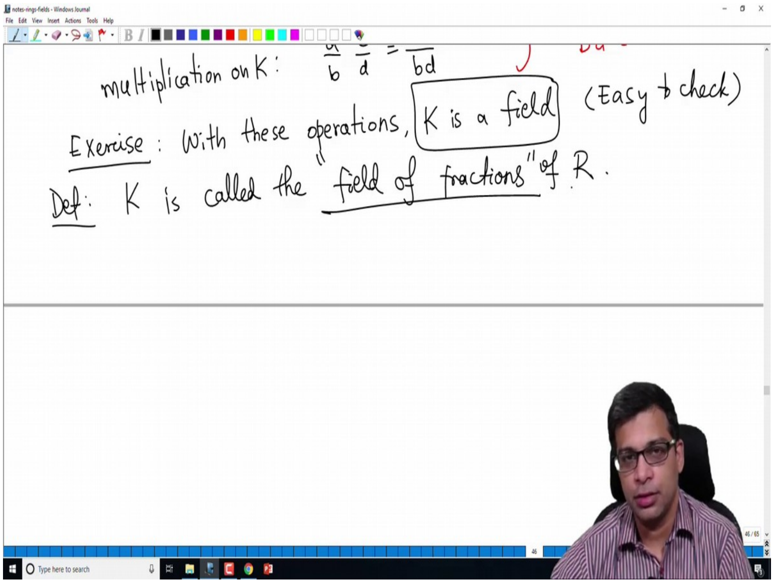

And this is a very easy exercise now to check that with these operations, K is in fact, not just a ring, but it is a field and this is an easy an exercise. So, I will not do this, I will let you check this, you can check that every element has under the addition, in fact, you have an abelian group under multiplication also every non 0 element has an inverse.

Certainly, because you take a by b, if a is non 0, b by a is also an element of K, so a by b times b by a is 1. So, its in fact, a field and this is an important definition now; K is called, the field of fractions of R ok. The name is very suggestive right, it is field of fractions means it contains all fractions of elements of R; in some sense it is the smallest field that contains R. Of course, if R is a field to begin with its fraction field is R, the R itself.

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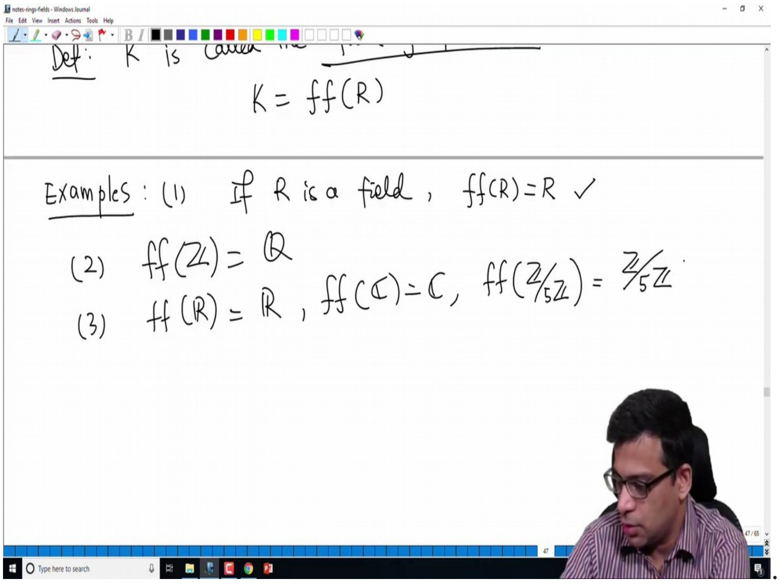

So, I will now write quickly some examples so 1, if R is a field, so let us denote the field of fractions by f f of R ok, just for simplicity, field of fractions, f f stands for field of fractions. If R is a field already R is a field what is the field of fraction of R? It is just R right, because you take fractions of elements of R, their elements of R again because R is a field a by b if a and b are in R 1 by b is in R. And b is non 0 then 1 by b is in R because R is a field.

So, a is in R one by b is in R, so a times 1 by b is in R. So, you get nothing new this is also confirms what I said, field of fractions is smallest field the smallest field containing a ring. So, if R is a field be to begin with you should not get anything bigger. So, it is field of fractions is itself and the model that I mentioned, field of fractions of Z is of course, Q.

And following the first example the field of fractions of R is R, field of fractions of complex numbers is complex numbers, field of fractions of Q is Q. What is field of fractions of Z mod 5 Z? Z mod 5 Z remember is already a field, so it is equal to itself.

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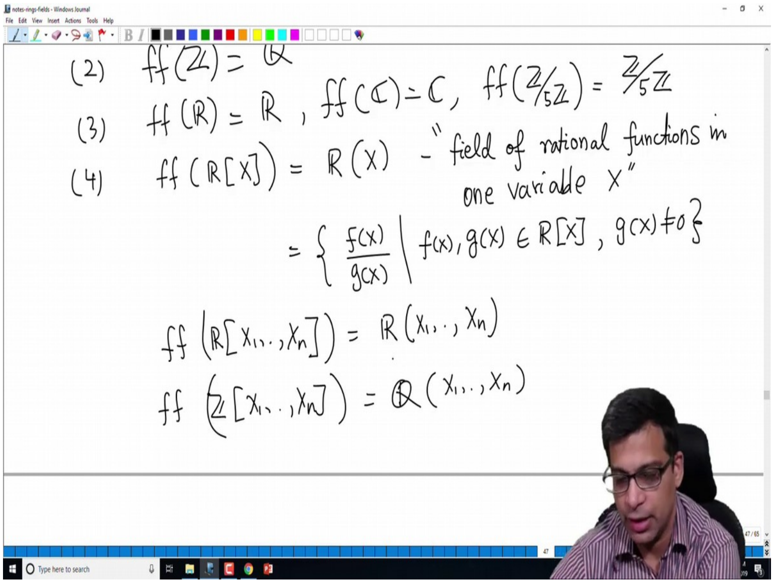

Some other examples, what is the field of fractions of, let us say polynomial ring in two variables. This is important for us maybe I should first do the one variable case. So, you take polynomials over one field R your real numbers. So, then this is what I will denote by R round bracket X. This is called field of this is the first time I am introducing this, this is a very important field this is called field of rational functions in one variable X ok. This consists of by the definition of fraction field all ratios of polynomials. So, f X g X are polynomials in R X with of course, g X non 0 ok.

So, actually I should go back to the definition of a fraction field, I should have mentioned this carefully. Remember that two fractions can be same, we say that a by b can equal c by d, if a d is equal to bc. So, we are not looking at all ratios or all fractions directly, we are looking at fractions under this equality. So, of course, this is something that you are familiar with right.

In the case of rational numbers, you have 2 by 4 is not considered a different fraction than 1 by 2, 2 by 4 is same as 1 by 2. So, you can in the case of a rational numbers you can cancel common factors. So, in general we want to insist that, I do not consider any fractions that we may be able to write as equal as unequal, if they are have this property that ad is equal to bc, I will consider them as same.

Similarly here any ratio I write f f X divided by g X will not give me a distinct element they could give me same elements. So, this is an important field for us. So, now, of course, you can figure out the notation for let us say I have, a polynomial ring in n variables what is its fraction field or field of fractions? It is simply I replace its square brackets by round brackets right, square brackets are replaced by round brackets.

And that of course, is not just the symbol, it carries a deep meaning which is that here I am looking at only polynomials in the polynomial ring, here I am looking at ratios of polynomials. Now what is the field of fractions of Z polynomial ring over the integers, of course, you have to put round brackets in other words you have to take ratios of polynomials.

But you also have to at the same time take ratios of integers because integers are already inside the polynomial ring as constant polynomials. So, you have to take Q Z square bracket X 1 through X n is Q round bracket, X 1 through X n.

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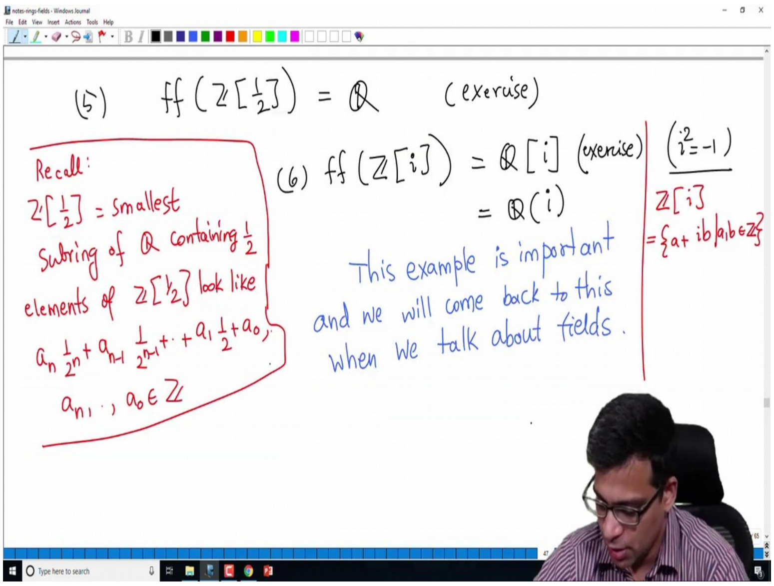

One more example: what is the fraction field of Z adjoined 1 by 2, remember Z adjoined 1 by 2 from one of the beginning videos in this course, so recall Z adjoined 1 by 2 is the smallest sub ring of Q containing 1 by 2.

Remember it already, any sub ring of Q contains, any sub ring of Q contains Z because a sub ring of Q is by definition a ring. So, it contains one; once it contains one it contains every integer n. So, every sub ring of Q contains Z, but it did not contain it need not contain 1 by 2, if it contains 1 by 2 we look at all sub rings that contain 1 by 2 and look at the smallest one. And explicitly elements of this ring look like you have to take essentially polynomials that is why we are using this square bracket notation to suggest polynomial rings.

So, we have a n 1 by 2 power n an minus 1 1 by 2 power n minus 1, think of this as polynomials in 1 by 2, where a 0 an through a n are integers so that is the ring. So, now, what, it is an integral domain because for example, it is it is contained in Q any sub ring of an integral domain is an integral domain. So, Q is a field, so its an integral domain so Z adjoined 1 by 2 is also an integral domain.

What is its field of fractions? Now you think about it a little bit you want to take add all ratios of elements of Z 1 by 2, but in particular you have to add ratios of elements of Z so that forces you to contain Q. And now you do not get any new elements by introducing ratio of 1 by 2. So, the fraction the field of fractions of Z adjoined 1 by 2 is Q. And I will finally, mention this example: what is the fraction field of Z adjoined i ok, Z adjoined i where i is a square root of minus 1 as always, Z adjoined i is simply defined exactly as far as the case of Z adjoined 1 by 2. Of course, now we have to define it to be the sub ring, the smallest sub ring of C the complex numbers containing i because we cannot take smallest sub ring of Q containing i because Q does not contain i.

So, what is a smallest sub ring of C that contains i, is precisely Z i. In fact, the as polynomials Z i has a much easier description, this is of the form a plus ib, where a and b are in Z. Because remember, we do not need to take all powers of i, i and i power of 0 are enough; because i squared is minus 1, i cubed is minus i, i power fourth is 1 and so on.

So, when we take polynomials, you just take this and what is the quotient fraction field? I claim that you have to of course take Q, because every you have to take fractions of elements of Z and then you can put either square bracket meaning all polynomials ok. So, or you can put round bracket this happen to be the same. So, now, I will mention that I will mention that this will be described in future.

So, these are equal; this example is important and we will come back to this, when we talk about fields ok. So, for now I will leave it as an exercise for you to convince yourself, that field of fractions of Z i is Q i, and once you add i I claim that 1 by i is also already there and 1 by i squared is already there and so on. So, that allows you to construct, explain why Q square bracket i is same as Q round bracket i.

But this is looking ahead; this is not that important if you do not understand this at this point, it is and we will come back to this. So, that is what I want to say for now about field of fractions. So, field of fractions is an important construction for us, especially when we talk about in the future when we talk about fields.

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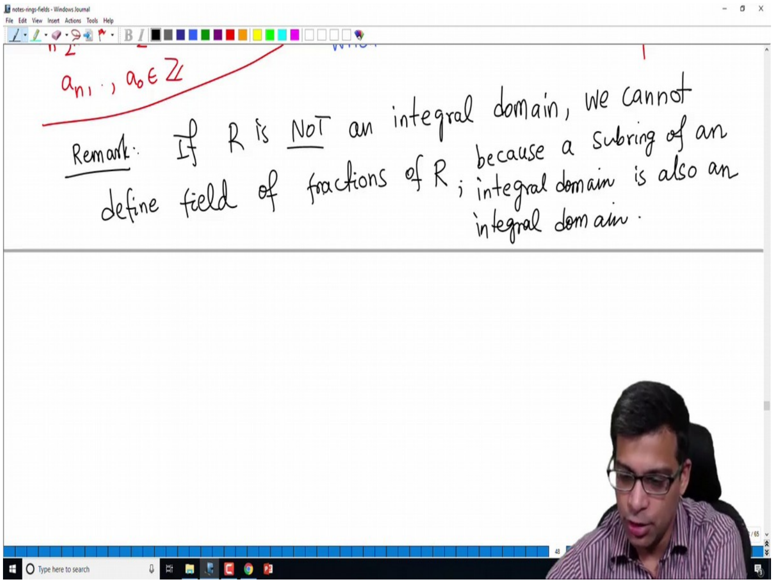

And actually one final comment I want to make is that if R is not an integral domain, we cannot define field of fractions of R right. So, of course, the definition that I have given for you earlier where I define all ratios or all fractions and add them or multiply them in this fashion clearly will not work because product of two non 0 elements in a non integral domain can be 0. So, if R is not an integral domain in other words there exists two element b comma d such that b and d are non 0 but the product is 0.

So, one by b and one by d you cannot define their sum or product, but maybe there is some other way to define field fractions, maybe there is some other way to put R inside a field, but this remark is saying that cannot be done. This is because a sub ring of an integral domain, is also an integral domain right.

Suppose you could put R inside a field, R is inside the field K; K is actually a field. So, it is a sub, it is an integral domain, R being a sub ring of K is an integral domain, but that of course, is a violation of the assumption that R is not an integral domain. So, if something is not an integral domain certainly we cannot define its field of fractions ok. So, that is so much about field of fractions.

Next I want to define a very important class of rings called “noetherian rings”.

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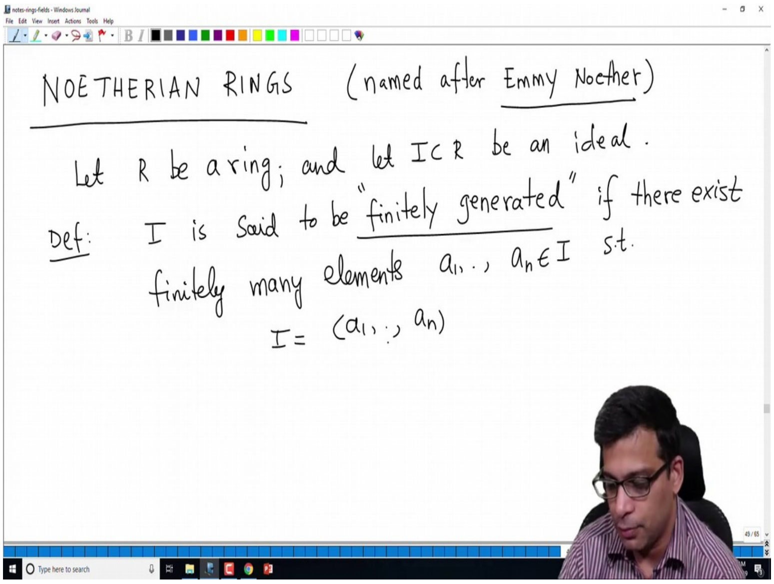

So, these rings are named after a mathematician called Emmy Noether who did very important fundamental work in commutative ring theory. So, now, noetherian rings are named after this mathematician who lived about 100 years ago ok. So, Emmy Noether did lot of work that we are discussing in this course ok.

So, let me go ahead and define, this is something that seems that would seem very natural after all these concepts that we have learnt. So, let us start with an arbitrary ring, so it is a good time for me to remind you that whenever I am talking about a ring in this course I am in a commutative ring with unity, multiplicative identity.

So, let R be a ring and let I be an idea of R. So, first let me define the notion of what it means for an ideal to be finitely generated; I is set to be finitely generated, if it is what the name is saying, if there exist finitely many elements, let us say a 1 a n in I such that I is equal to the ideal generated by an. So, I have already discussed in some previous video, what it means for a collection of elements to generate an ideal.

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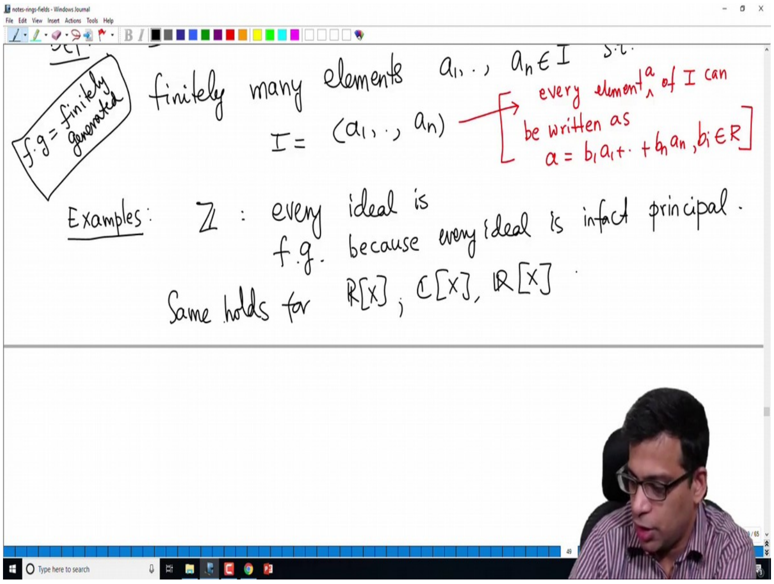

So, this simply means that, now this means that every element of I can be written as some linear combination of a 1 through an. So, every element let us say a if I can be written as a equals b 1 a 1, where b I is are in R ok.

So, you are allowed to take coefficients from the ring, but it must be a linear combination of a 1 to through an. Remember a 1 squared a 1 cubed will also be there, but we do not need to include squares here or any other powers because you can absorb any other extra powers of a 1 into the coefficient, because its an ideal we can always write every element like this. So, finitely generated simply means that there is a finite set of elements which generate I, meaning every element of I can be written as a linear combination of this ok.

So, examples are plenty, because almost everything that we have seen so far. In fact, probably everything that you have seen so far is an example of finitely generated ideal. For example, if you take Z, every ideal is finitely generated right. So, I am going to write fg is short for finitely generated ok, every ideal is finitely generated in Z because, in fact, it is principal, not just finitely generated right.

There is a stronger statement that is true, every ideal is generated by a single element we only need in fact, that every ideal must be finitely generated I am saying, but even better is true in fact, every ideal is principal. Similarly same holds, the polynomial ring over in over a field in one variable ok.

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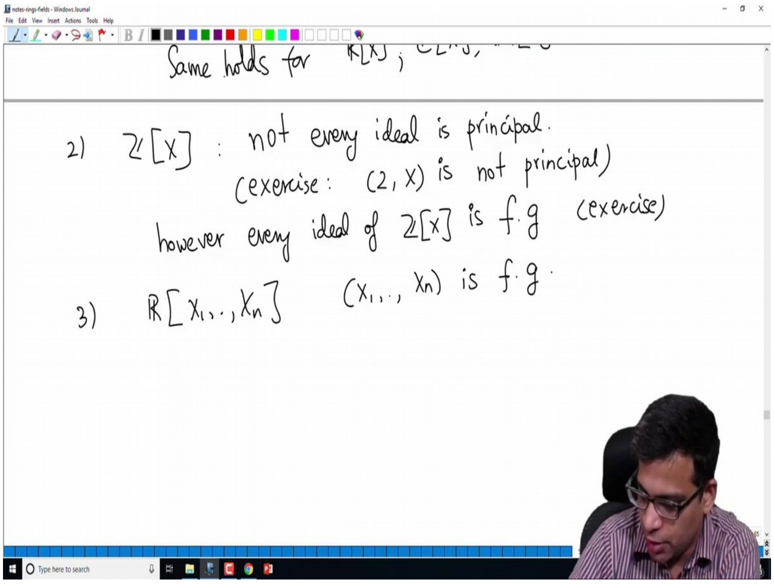

So, here also every ideal is principal so every ideal is finitely generated in particular. On the other hand if you take polynomial ring over Z here not every ideal is principal, this I did not ever mention explicitly before and this is an exercise now to check for you. The ideal generated by 2 X, 2 comma X is not principal there cannot be a single element which generates 2 comma X.

Because you take a polynomial of non constant polynomial it will not be able to generate 2, if you just take a constant if you take a constant polynomial it will not generate X ok. So, no single element will generate the ideal which is generated by 2 comma X. However, every ideal of Z X is principal is finitely generated. So, this is also an exercise for you, it need not be generated by a single element, but it is generated by finitely many elements, in fact, I claim that it is generated by two elements, you can check this.

So, what you need to do is take the smallest positive integer that is contained in a given ideal and argue that that and X will generated. So, this I will leave for you for that and some polynomial will generate it I should say ok. So, this is an exercise may be I will come back to this later.

NOTE: THE ABOVE ASSERTION IS FALSE; SEE THE CORRECTION IN THE NEXT VIDEO.

So, this is another interesting example, if you take on, other hand polynomial rings in several variables, here we cannot, let us make any statement about all ideals, we will do that in a future video, but I can talk about this particular ideal is finitely generated ok. In fact, what is true in this ring is that, every ideal is finitely generated that I will mention next time.

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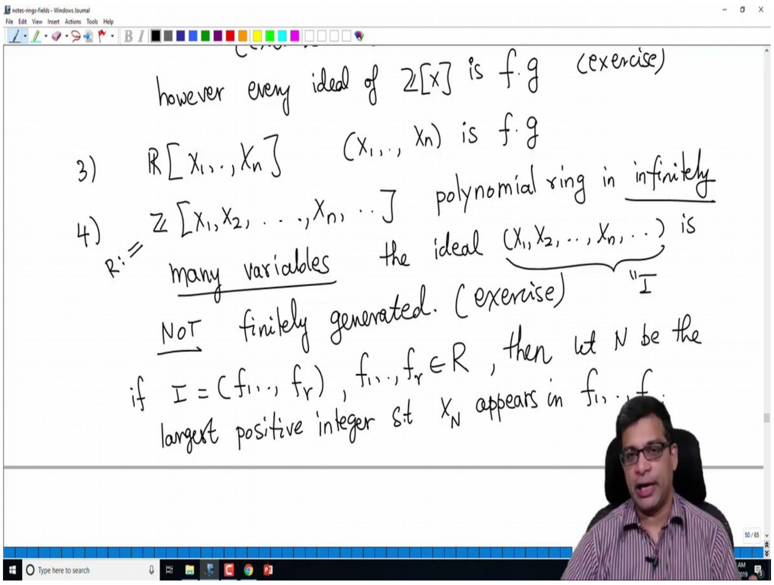

On other hand if you take, a polynomial ring in infinitely many variables this we have not encountered before. And I should define this carefully I will only say this orally for now, we will not really discuss this example a lot in this course, but what is a polynomial ring in infinitely many variables. This any polynomial is actually a finite sum, but it can take variables from this infinite countable infinite collection of variables ok. So, then you can naturally define addition multiplication just you like you define in the case of finite number of variables.

In this the ideal generated by all the variables is not finitely generated. So, this is a good example of a non finitely generated ideal. So, why is this not finitely generated? This is a simple exercise again I will leave for you; suppose if it is finitely generated, it is generated by let us call this ideal I, if I is finitely generated let us say generated by some r polynomials and let us call this ring R where f 1 through f r are in capital R.

Then, let N capital N be the largest positive integer such that the variable X N appears in f 1 through f r. So, why is that such a capital N exists? That is because remember each polynomial, though we are working in the working inside the polynomial ring in infinitely many variables a specific element is a finite polynomial. So, it can have only finitely many terms, so it may have X 1 million X billion and so on, but it must stop somewhere.

So, f 1 is a finite sum of terms, so you can figure out the largest index of the variable that appears in f 1, the largest index of the variable that appears in f 2 and similarly the largest index of the variable that appears in fr and pick the largest one and call that capital N. So, a capital X capital N is the largest index of a variable that appears in a f 1 through fr.

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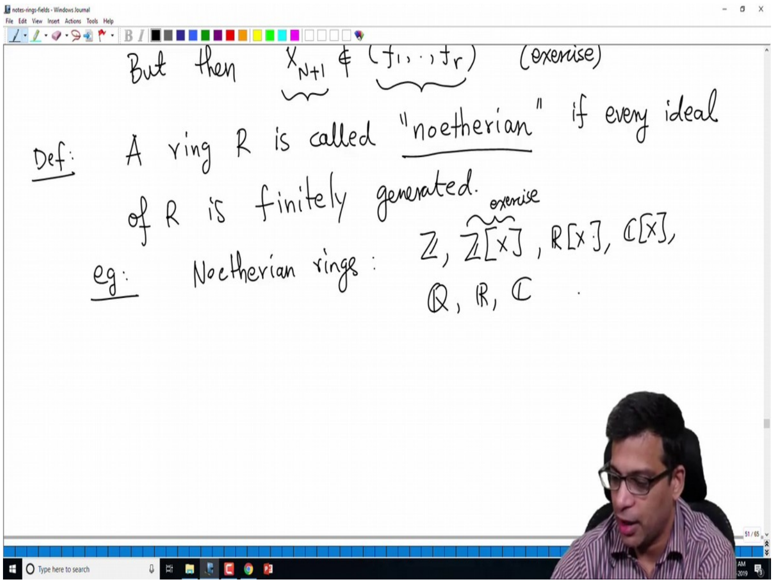

But now because we have infinitely many variables we can consider the next index. So, then X N plus 1 cannot be in the ideal generated by f 1 through f r. Because remember, when you take and the element of this ideal; that means, you take a 1 f 1 plus a r f r, where a 1 through fr are elements of R you cannot introduce a new variable, you can introduce a new variable, but it cannot just be a single variable ok. So, this last part is somewhat a subtle exercise, but I will leave that for you it cannot be a single variable.

So, in other words f 1 through f r cannot generate X, the ideal I. So, however many you take, if you take only finitely many that finite set of elements cannot generate the ideal I. So, it is not finitely generated ok. So, now, let me end this video by giving you the main definition that I want to discuss in the next video.

A ring R is called "noetherian" ok, so I will use this word to describe rings where every ideal is finitely generated, if every ideal of R is finitely generated then I say that R is noetherian. So, quickly let me some examples, what are rings? Examples of Noetherian rings, almost every ring that we have considered so far is noetherian, sometimes it is in some cases it is trivial to check this.

Of course, Q, R, C, are also noetherian, because Q, R, R, C; Q, R, C are fields so only two ideals, the unit ideal and the 0 ideal which are of course, finitely generated. Every ideal in Z is principal every ideal in Z X, I said is finitely generated, so that is actually an exercise. Every ideal in R X is principal ideal, every ideal in C X is a principal ideal which you can check by doing Euclidean division. So, these are all rings and we will in fact, check that polynomial rings in finitely many variables are noetherian later.

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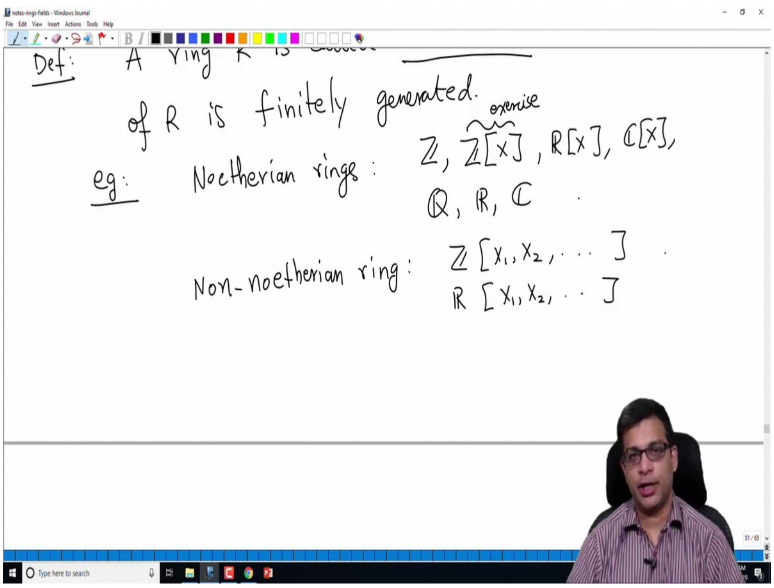

And what is an example of non noetherian ring, something that is not noetherian. The only example we know is the polynomial ring in infinitely many variables which of course, I can put coefficients to be anything I want. So, these are not noetherian, because I have said here that the ideal generated by all the variables is not finitely generated and we will see one more example of a non noetherian ring later.

But non noetherian rings are very rare, so that is a take away from this and the most interesting examples that we will study in this course are noetherian. So, let me stop the video here, in the next video we will continue our study of noetherian rings.

Thank you.