Introduction To Rings And Fields
Prof. Krishna Hanumanthu
Department of Mathematics
Chennai Mathematical Institute

Lecture - 16
Existence of maximal ideals

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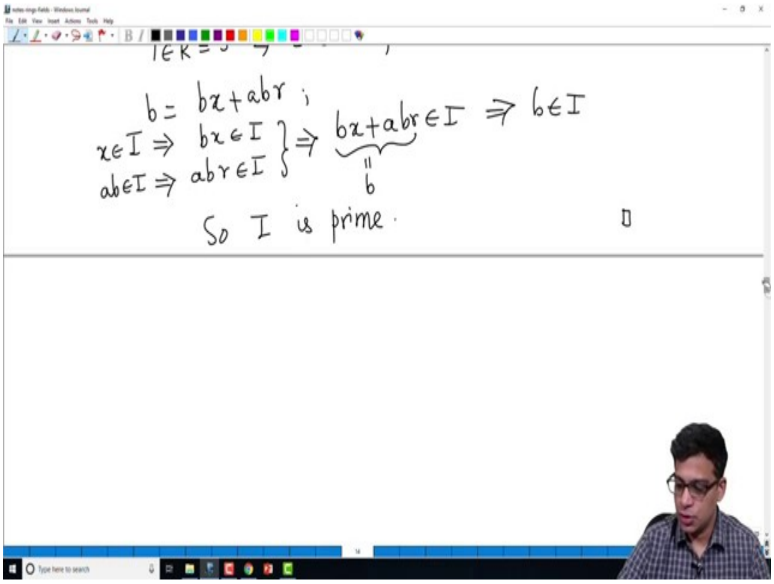

Let us continue now. In the last video we looked at maximal ideals. So, I am going to continue discussing them and also solve some problems in this video. So, remember maximal ideals are ideals, which have no ideals that contains them properly and those ideals are proper. So, if any ideal contains them properly then it must be the full ring ok.

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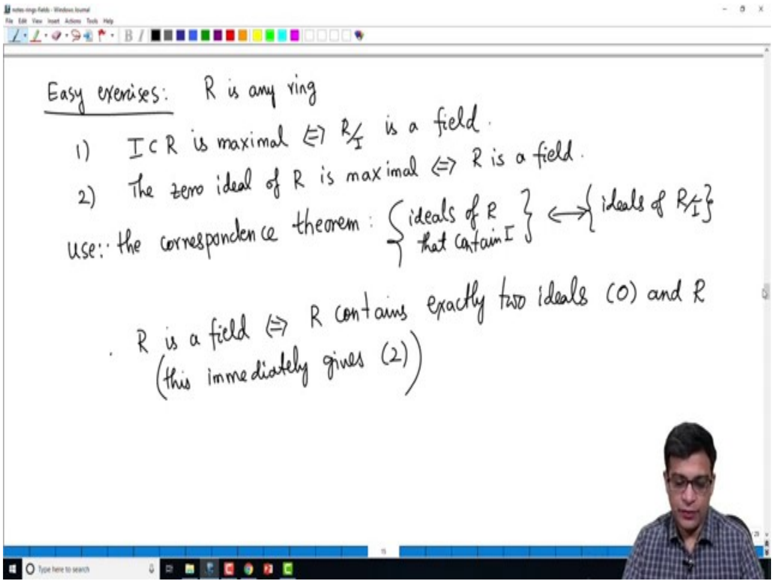

So, then obvious easy exercises for you some to begin with, just like we had the notion of integral domains which characterize prime ideals, there is an existing notion which characterizes maximal ideals which is exactly the notion of fields.

So, the point is so the two exercises I will write here the zero ideal, so in fact, let me write it like this I is an ideal which is maximal. So, R is any ring here. So, I is an ideal which is given to you it is maximal if and only if R mod I is a field. So, in so let me tell you how to solve this and as a consequence of this let me make another statement first as a consequence of this. The zero ideal of R is maximal if and only if R is a field; so this can be done more directly. And in fact, both problems are a consequence of the correspondence theorem that we have done.

So, use correspondence theorem which tells you how ideals of R that contain I are in bijective correspondence with ideals of R mod I right, this is the theorem that we have proved. If you simply take the canonical natural map from R to R mod I and you look at the image of ideal in R in R mod I this gives you a bijective correspondence between ideals of R that contain I and ideals of R mod I. And we have to use this and also this fact that. So, this is first fact other fact is R is a field this also I have mentioned before, if and only if R contains exactly two ideals R is a field if it contains exactly two ideals; namely the 0 ideal and R remember every ring contains two ideals 0 and R it is a field if and only if these are the only ideals.

So, this immediately tells you how to solve problem 2 right, because if R is a field then 0 does not have any ideal that contains it other than the full ring and the 0 ideal also so, it is certainly a field. Similarly if it is a field if the 0 ideal is maximal; that means, there is no other ideal other than 0 and R. So, this immediately gives 2 right this immediately gives 2. And in fact, now using the correspondence theorem you get 1 there is no problem now because, R mod I is a field by 2 if and only if 0 ideal of R mod I is maximal, but 0 ideal of R mod I is maximal if and only if I contains no ideal or no ideal contains I. So, basically maybe I should explain this in more detail.

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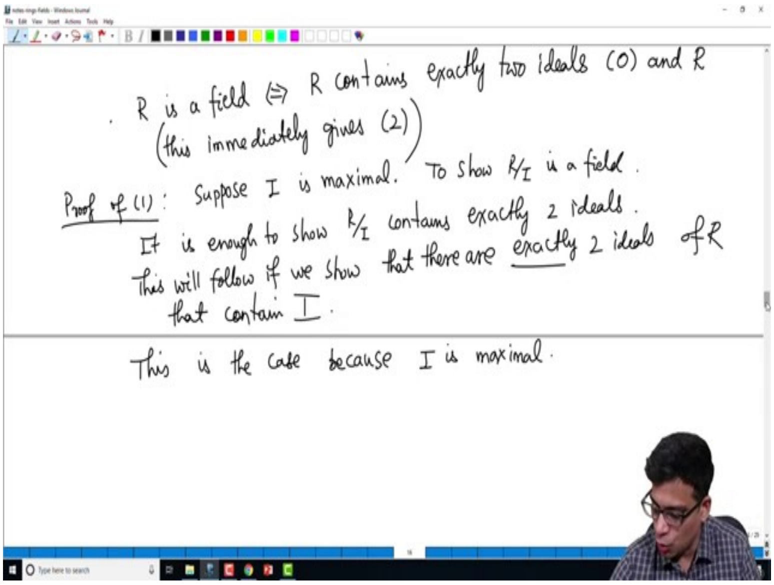

So, proof of 1, solution of 1. So, remember what are we trying to show R is I is maximal if and only if and only if R is R mod I is field. So, suppose I is maximal, suppose I is maximal then, then consider so to show you want to show R mod I is a field. Now it is enough to show by the property I wrote earlier it is enough to show R mod I contains exactly 2 ideals. In other words this set here, ideals of R mod I is exactly 2 elements set namely the 0 ideal of R mod I and the unit ideal of R mod I, but in other words we want to show that ideals of R that contain I are also exactly 2, but this is clear now.

This will follow, if we show that there are exactly 2 ideals of R that contain I, this is true by definition of maximal ideals this is true because I is maximal right. So, the only 2 ideals that contain I, remember there is another way of saying something is a maximal ideal the only 2 ideals that contain a maximal ideals are maximal ideal are the 0 ideal the that ideal itself and the unit ideal.

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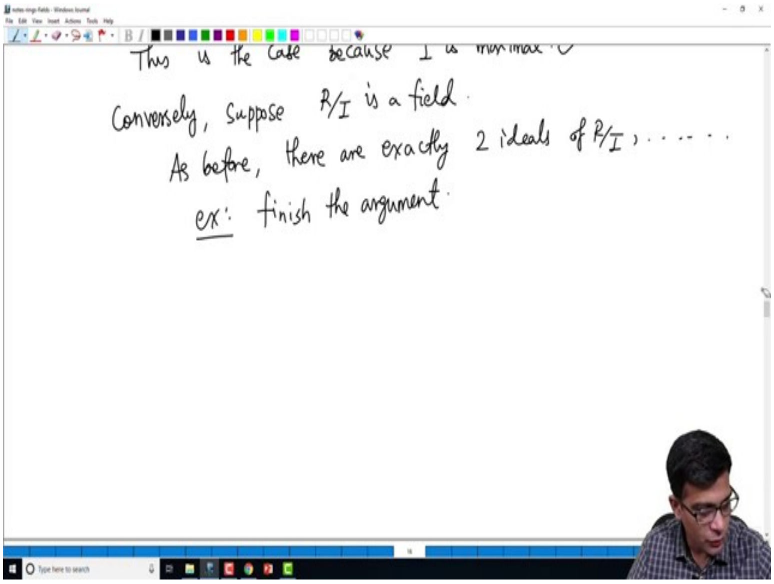

So, that is true hence there exist exactly 2 ideals that contain I in R hence their images which are the 0 ideal and R mod I are the only 2 ideals in R mod I. So, R mod I is a field. So, if I is maximal R mod I is a field conversely suppose, R mod I is a field. Again there are exactly as before, so this I will leave for you just start the solution and I will leave it for you. There are exactly 2 ideals of R mod I because, R mod I is a field only ideals are 0 and unit ideal.

So, there are exactly 2 ideals of R mod I now we know that this set has 2 elements, so this set ideals of R containing I has only 2 elements namely I and R so I is maximal. So, I will finish I will let you finish this exercise, finish the argument in fact, I have done it, but if it is not clear think about argument and finish it.

So, this is a way to characterize, maximal ideals of rings just like we knew how to characterize prime ideals of rings remember are those ideals whose quotient ring is integral domain. Here maximal ideals are those ideals, whose quotient ring is field ok. So, this tells you how to compute how to find max whether an ideal is maximal ideal or not ok.

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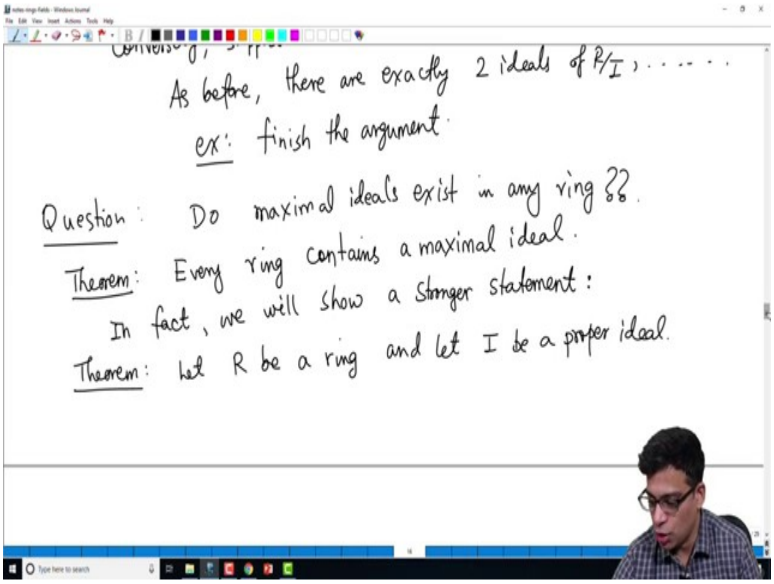

So, now, so let me look at some other exercises, before that the question that we should ask even before doing any exercises is do prime ideals or maximal ideals exist, do maximal ideals exist in any ring right. So, do maximal ideals exist in any ring? If you show that they do then certainly prime ideals also exist, because we know from the last video that maximal ideals are prime.

So, and this is a theorem, which uses something called Zorn's lemma which may be you are familiar with. So, I am going to quickly recall it and prove it, but this is not really an important proof the statement is important every ring remember rings for us are always commutative rings with unit, so every ring contains a maximal ideal. In fact, we will show the following, so in fact, we will show a stronger result, what is the stronger statement?

Theorem: Let R be any ring and let I be a proper ideal, proper ideal remember means it is not equal to R it can be 0, but it is not equal to R.

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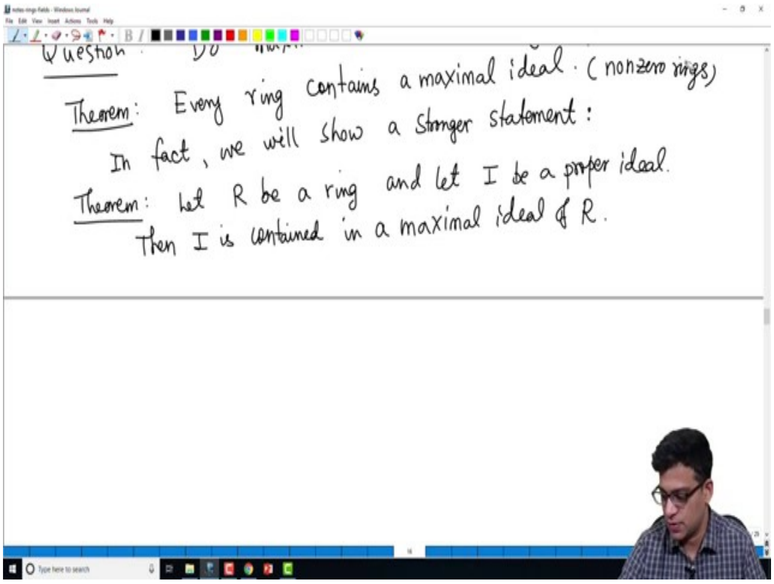

So, let I be a proper ideal, then I is contained in a maximal ideal of R right. So, I am saying that every proper ideal is contained in a maximal ideal this will show that every ring contains a maximal ideal. So, this is because so of course, I assume that nonzero rings. So, I am always going to consider nonzero rings. So, every nonzero ring contains a 0 ideal of course, and the 0 ideal is a proper ideal, because the ring is not the 0 ring.

So, now, by the above theorem the 0 ideal is contained in a maximal ideal. So, there are maximal ideals. So, why is this? So, think about how to prove such a statement. Suppose you are given an ideal I you want to show that it is a max it is contained in a maximal ideal so how do we start?

So, suppose I is maximal then you are done because I is a maximal ideal and I contains I. So, I is contained in a maximal ideal if I is maximal, you are done. Suppose I is not maximal, then by definition of not being maximal I is contained in a bigger ideal may not be maximal ideal, but I is contained in a bigger ideal.

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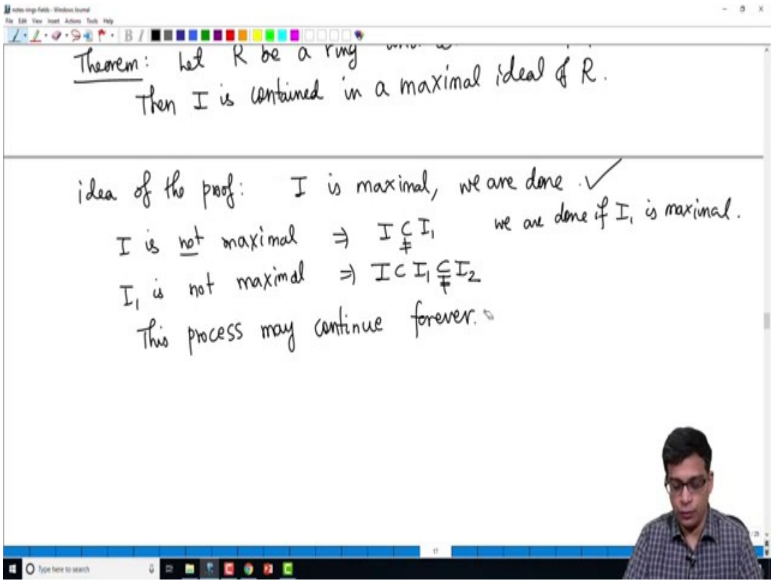

So, this is the idea of the proof. So, if I is maximal we are done right. Because then certainly I is contained in a maximal ideal suppose I is not maximal. So, if I is not maximal, implies I is contained in J right by definition of a non-maximal ideal it is contained in a bigger ideal. So, J if it is maximal we are done, so we are done. So, let us actually call it some I 1, we are done if I 1 is maximal ok, because I is then contained in a maximal ideal.

If not, if I 1 is not maximal then I 1, so we have I in contained in I 1 which is contained in I 2. So, I 2 is a bigger ideal, if I 2 is maximal then we are done otherwise you continue. So, you can keep doing this and you think that at some point you will stop and get a maximal ideal. Now in mathematics this is a certain point you cannot argue that may be it will always stop may be it forever continues why do we know that there is a point at which we can say, that this process ends and we do get a maximal ideal. So, there is something called Zorn's lemma, so this process may continue forever right. So, there is we have no reason to assume that it does not continue forever.

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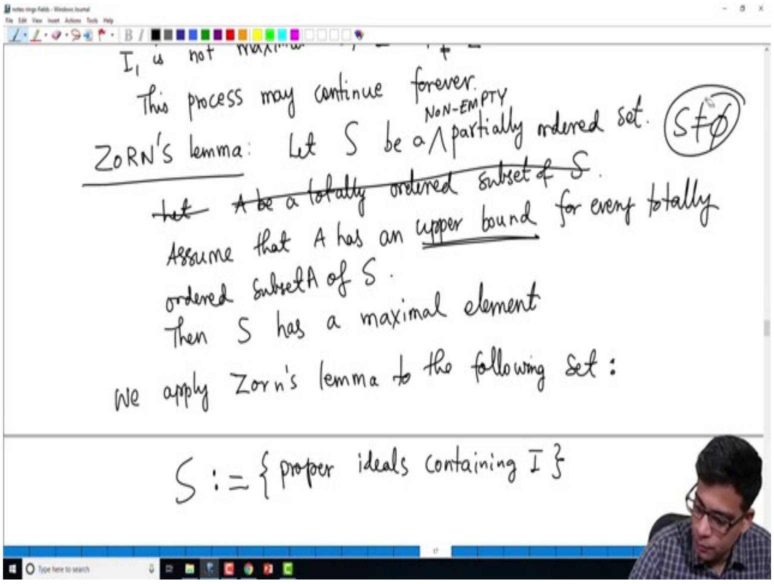

So, there is a tool that we use called Zorn's lemma which is going to tell us that we can go around this problem and get the maximal ideal that we want. Zorn's lemma, so I do not want to now at from this point in the next 10 minutes I will do some I will recall what is Zorn's lemma and I will tell you how to use it to show the theorem that every proper ideal is contained in maximal ideal, but this is only for people who have seen some of these things before; and if not this is maybe you can listen to it, but it is not important. The main statement is the theorem and the statement that every ring contains maximal ideal is something, we will know we will useful we will use later the proof is not that important, so if it is not clear to you, do not worry about it.

Zorn's lemma says that, so I am going to use some notion from set theory. So, S let S be a partially ordered set so; that means, there is an order on this, but it is a partial order; that means, maybe not every pair of elements can be compared. In that, let A be a totally ordered subset of S; that means, every 2 element of S can be compared and say you can say one is bigger than the other; S is only partially ordered, but A is a totally ordered set ok. In fact, I should may be say that assume that A has an upper bound ok, so the statement is that A has an upper bound for every so, I should not really write I should really say assume that A has an upper bound for every totally ordered subset of A of S.

So, S is a partially ordered set in which every totally ordered subset has an upper bound; that means, there is an element of S which is bigger than everything in S, that is what an upper bound is, then S has a maximal element; this is Zorn's lemma. So, again I will remind you this is only if you know a little bit of this and it makes sense to you otherwise this is not really part of ring theory, so do not worry about it; S has a maximal element. So, we will use this lemma this is an axiom in mathematics that you assume that this is true.

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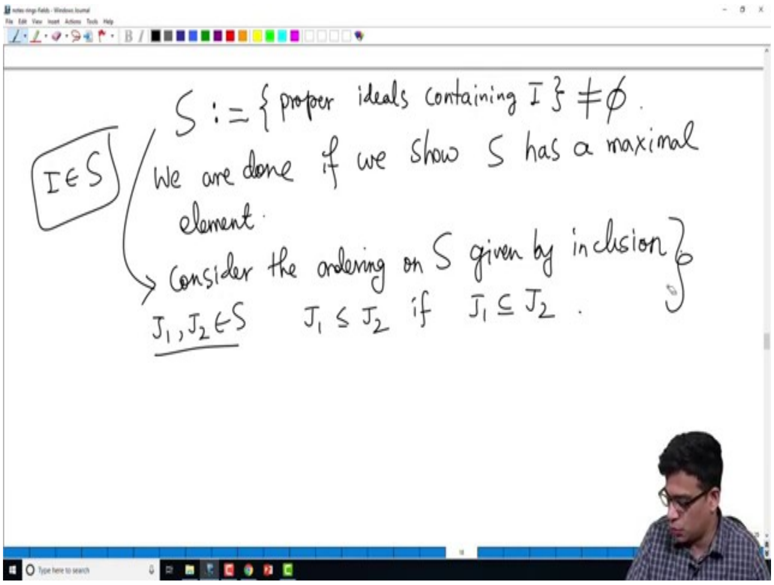

So, now we apply Zorn's lemma to the following set. So, S to be, to be the set, ideals containing I proper ideals. So, I should write that proper ideals containing. So, I forgot an important assumption in Zorn's lemma, it is a non empty partially ordered set.

So, S is not empty ok. So, we otherwise there is no statement you can make that S has a maximal element empty set has no maximal element. So, we need S to be non empty. So, certainly in now in this application S is non empty because I has a proper ideal by hypothesis it contains I, so I belongs to S.

So, I am considering all ideals that are proper and that contain S. And we are done if we show S has a maximal element why is that? Before that I should really say, so this should come before actually give consider the ordering given by consider the ordering on S given by inclusion. So, in other words we say that I so some 2 ideals in, so let us say J 1 J 2 are in S will say that J 1 is less than equal to J 2, if J 1 is contained in J 2.

So, this is how you give it a partial order remember this is only a partial order because, not any two not every pair of ideals may be compared may be neither contains the other so, but it is a partial order we can compare them and this ordering has the usual properties. So, now, if we show that S has a maximal element with respect to this partial order we are done because, if S has a maximal element that must be a maximal ideal of the ring because if not there is something bigger that contains set.

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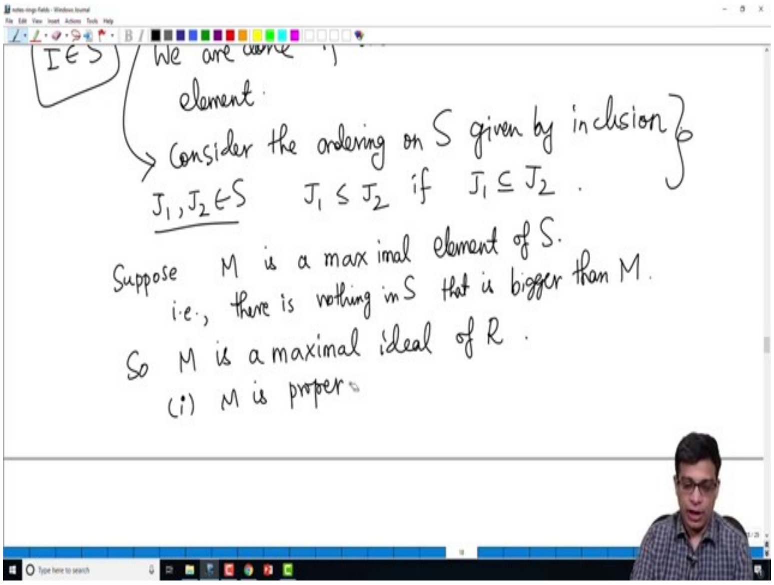

So, suppose M is a maximal element of S; that means, there is nothing in S that is bigger than M, that is, there is nothing I will just write it informally like this there is nothing in S that is bigger than M right.

So, so M is a maximal ideal of the ring itself, why is this; first of all first M is proper right, that is clear because M belongs to S and S remember is the set of proper ideals that contain I.

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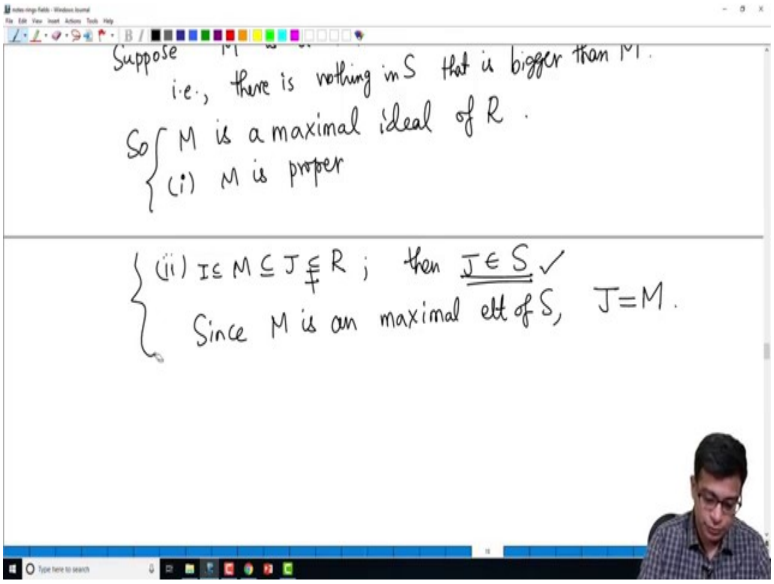

So, it is proper, so suppose M is contained in J and J is contained in R then remember J is also in S. So, suppose J is not equal to R, suppose J is not equal to R we will show that J is equal to, if J is not equal to R then J belongs to S because J is a proper ideal remember I contains M, I is contained in M. So, J is contained in M sorry, I is contained in M, M is contained in J, so J contains I. So, and J is a proper ideal, so J is in S, but since M is maximal element of maximal element of S and J is bigger than M possibly, but it cannot be right, so J is equal to M.

So, if we show that S has a maximal element then we are done and this is where Zorn's lemma comes in Zorn's lemma says that, gives you under some conditions a partially ordered set has a maximal element; what is that condition? Every totally ordered subset must have an upper bound.

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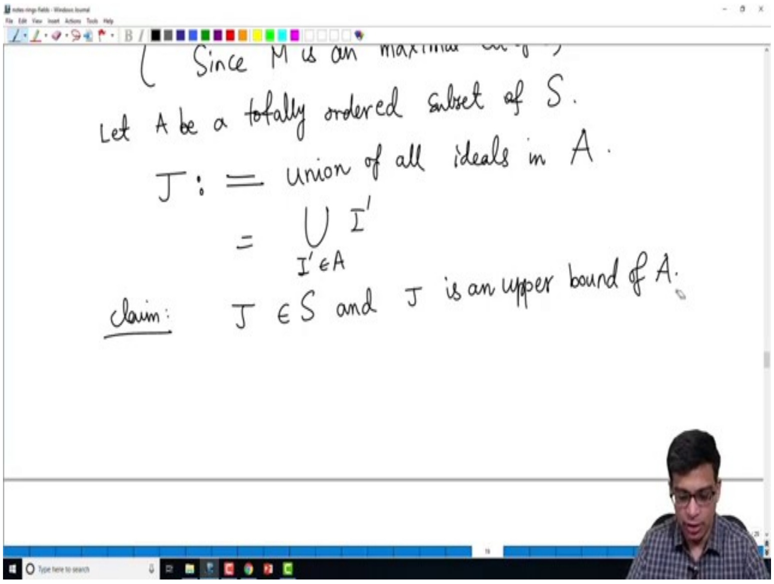

So, let us take a totally ordered subset, so let A be a totally ordered subset of S. So, we want to show that A has an upper bound then we are done then Zorn's lemma says that, S has a maximal element and we argued that any maximal element of S is in fact, a maximal ideal of the ring. So, let A be a totally ordered subset of S. Now I claim that consider this ideal J, define it to be union of all ideals in A. So, I take J to be union I prime where I prime is in A.

So, now claim is J is in S and J is an upper bound of, I am working with an arbitrary partially ordered set sorry arbitrary totally ordered subset A of our set S. To apply Zorn's lemma I need to show that A has an upper bound and my claim is that this J that I defined is an upper bound. So, I need to prove that J first of all belongs to S.

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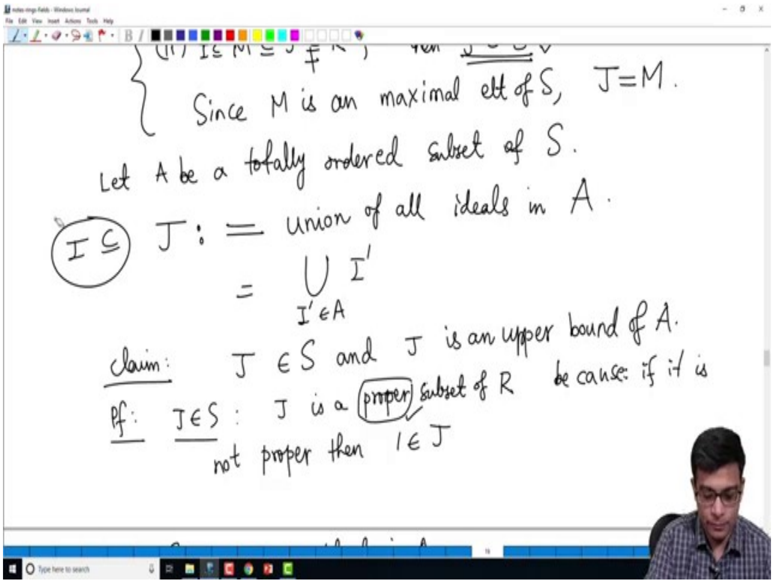

So, what is S? If you recall S is the set of proper ideals that contain I. So, we will first show that J is an S, first of all J is a proper ideal of R, why is this, J is a proper ideal of R because if it is not if it is not proper what is the meaning of not being proper if it is not proper then 1 belongs to J right; that means, J is the unit ideal. So, one belongs to J, but J is the union set theoretic union of ideals of A.

So, so 1 belongs to some ideal of contain in A. So, that is there exists I prime in A such that 1 is in A prime, but this is a contradiction right, why is that? This is absurd because, A only contains proper ideals by definition S contains only proper ideals, so A contains only proper ideals. So, in particular I is I prime is proper, so I prime cannot be cannot contain 1. So, J is a proper ideal sorry actually we do not yet know that it is an ideal. So, I will say that J is a proper subset, I should say of R, J is a proper subset of R because if it is not then it is equal to R then it contains 1 and we showed that that cannot be.

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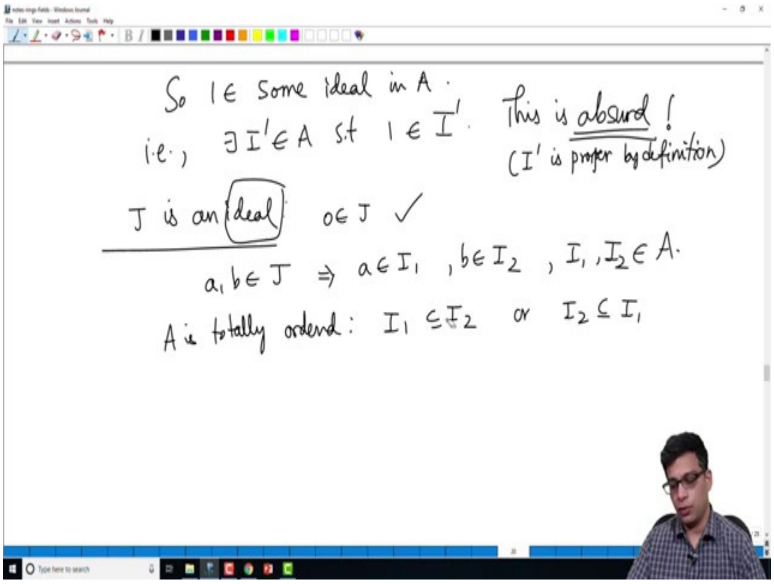

Second point is J is an ideal, so we have three words here: in S it is a proper ideal contains I. So, we showed that it is proper we will now show it is an ideal then we will show it contains I. So, J is proper that is correct now we show that J is an ideal why is J an ideal? This is the most interesting thing and in fact, this is where we need that A is capital A is totally ordered. So, remember what is an ideal, so if 0 certainly belongs to J because 0 belongs to every element whose union is J, I prime. So, now, on the other hand suppose a, b are in J we want to show that a plus b is in J if a, b is in J then a is in I 1 and b is in I 2, where I 1, I 2 are in A right.

A is a collection of ideals union of all the ideals that are contained in A is J. So, if 2 elements are in if an element is in J it must be one of them one of the ideals of A, because J is the union of all ideals of A. But now since A is totally ordered, I 1 is contained in I 2 or I 2 is contained in I 1, this is not true for a partial ordered set, but true for a totally ordered set.

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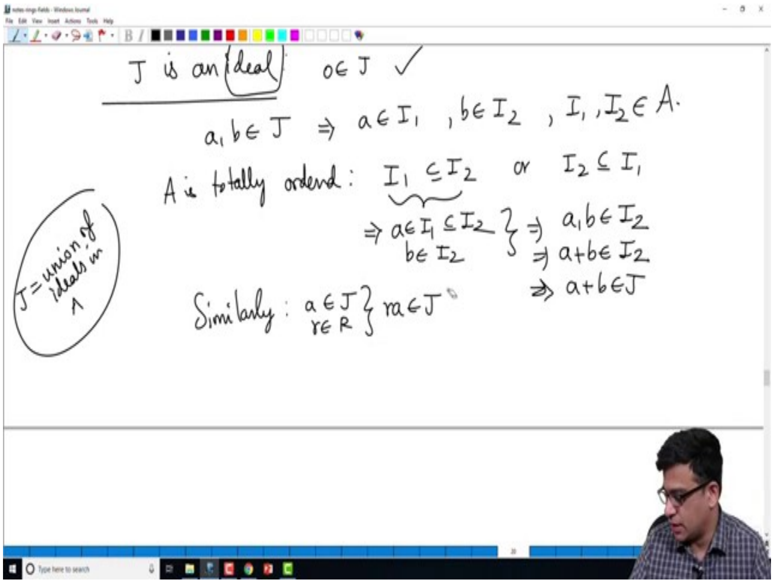

So, suppose this happens, without loss of generality, this happens then a belongs to I 1, which is contained in I 2, b belongs to I 2; that means, a, b belong to I 2, but I 2 is an ideal; that means, a is in a plus b is in I 2, but; that means, a plus b is in J because I 2 is contained in J, J is the union of ideals in I will write it again so that you can see J is the union of ideals.

So, I 2 is an ideal of a plus b is an I 2, so a plus b is in J. Similarly, you can conclude this is more easy if a is in J and r is in R then ra is in J so J is an ideal that is ok.

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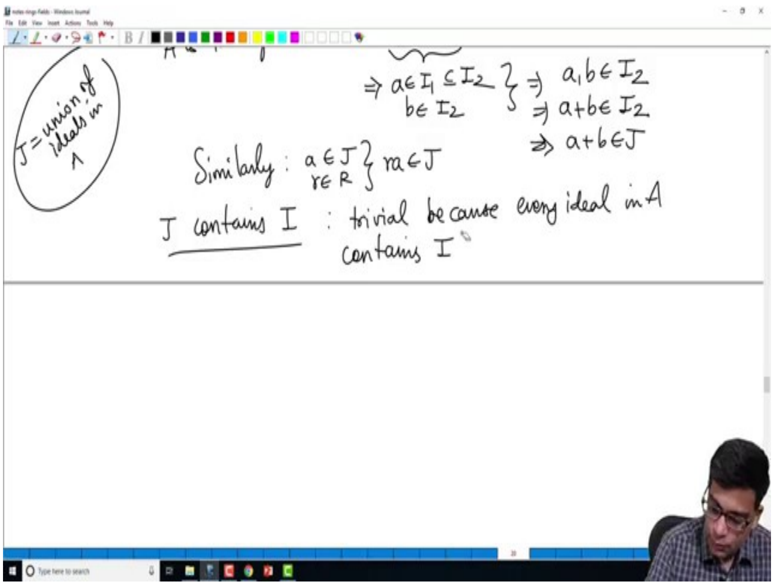

So, and finally, J contains I trivially right this is trivial why is this? Because every ideal in A contains I by definition, right because our whole set capital S is things that contain I and A is a subset of S. So, every element of A contains I their union also contains I. So, J is in fact, going to trivially contain I so what did I show now, J is in S and so what I have shown is that J is in S is what we have shown.

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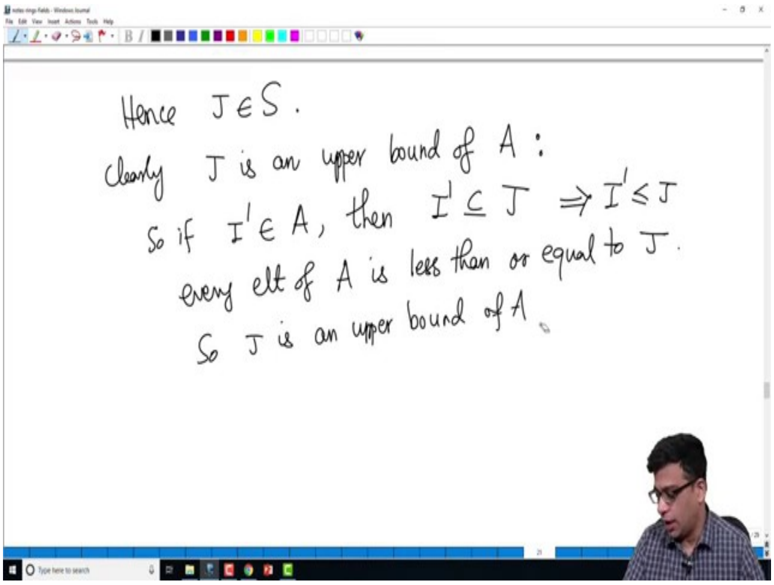

Hence J is in S is an element of S remember, S is the collection of ideals J is an ideal, so J contains, J is contained in S.

And clearly this is very easy, J is an upper bound of A because remember J is the union of all ideals in A. So, if I prime is in A then I prime is contained in J because J is the union of all ideals in A in particular I prime is one of the things that contributes to that union. So, J contains I prime; that means, I prime is by definition of the order on S, I prime is less than or equal to J. So, every element of A is less than or equal to may be it is equal to, you don't need to know about that, less than equal to J. So, J is an upper bound.

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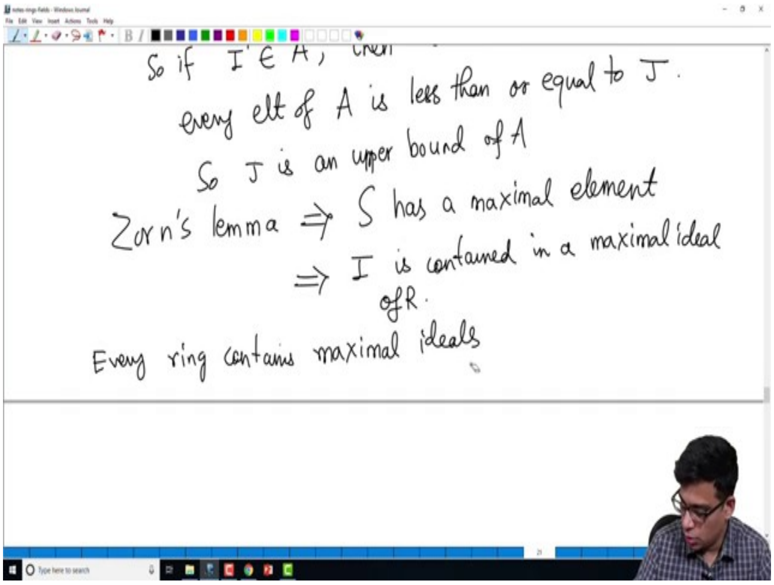

So, in other words we have verified the hypothesis of Zorn's lemma. Now, Zorn's lemma says that A, sorry S has an upper bound, has a maximal element, it is not an upper bound of S remember I am not saying that every element of S is less than that maximal element. I am saying that, there is nothing bigger than that element in S, in other words or I is contained in A, this I argued earlier I is contained in a maximal ideal of R.

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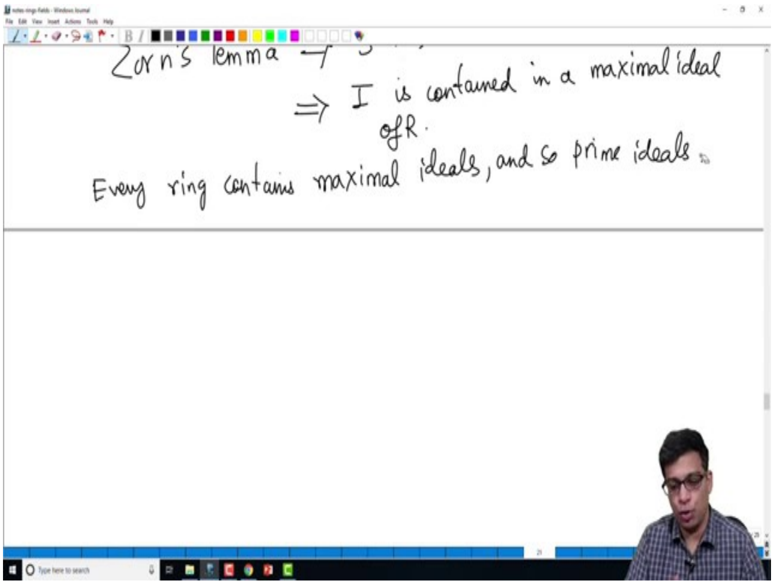

So, in conclusion every ring contains maximal ideals, and hence and so prime ideals. So, this is good to know so because maximal ideals are prime. So, every ring contains maximal ideals I have shown, hence every ring also contains prime ideals.

So, now what I will, so I will end the video here, but let me just remind you again that the upshot of this video is that, we have learnt how to characterize maximal ideals; these are ideals such that the corresponding quotient ring is a field. And similarly in any commutative ring, something is 0 ideal is maximal if and only if the commutative ring is a field; and rest of the video we spent proving that every ring contains maximal ideal maximal ideals.

So, this now, proof itself is not that important, if the proof is not clear, you can think about it and you can ask questions, but this is not required for rest of the course. The statement is only required that every ideal, every ring contains maximal ideals is needed, but the proof is not needed.

Thank you.