Introduction To Rings And Fields
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Lecture – 15
Maximal ideals, integral domains

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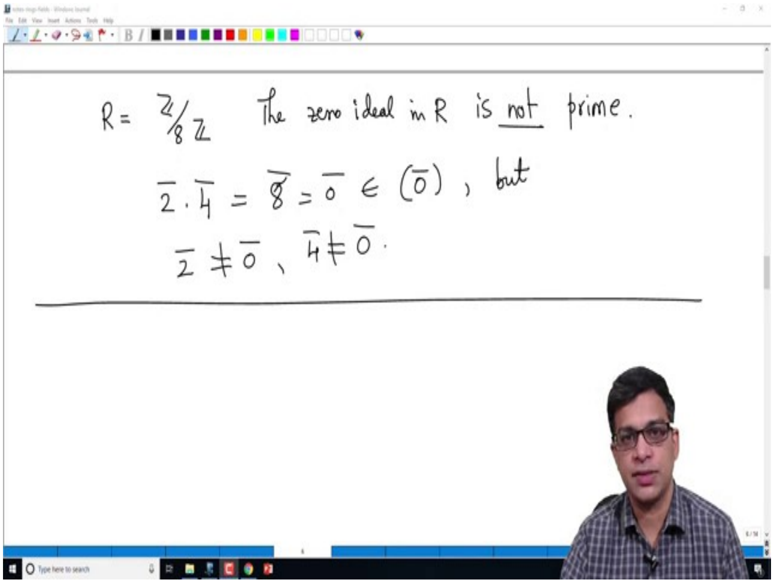

Let us continue now, in the last video we looked at prime ideals in a ring these are ideals where if product of two elements belongs to the ideal, one of the elements belongs to the ideal.

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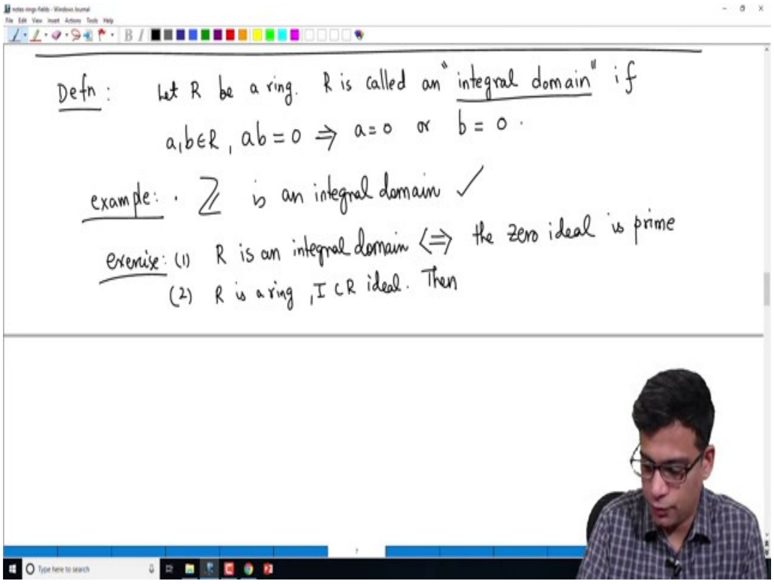

so let us now continue. So, I am going to define an important class of rings now. So, this is the definition that I will give now. So, let R be a ring; so, R is a ring. So, R is called an "integral domain". So, these are important class; this is an important class of rings R is called an integral domain if the following property holds: a, b in R if a comma; a times b is 0, ab means a times b, 0 that implies that a is 0 or b is 0.

So, if you have so, simply put, it says that if you have two elements in the ring whose product is 0, one of them must be 0. This is a very familiar property for us for example, integers have this property. So, immediate example the most obvious example is Z is an integral domain. This is because, if you multiply two integers and you get 0, one of the integers must be 0, if both are non-zero, their product is also non-zero. So, Z is an integral domain, this is easy. And in fact, this also suggests to you that this is related to the definition of prime ideal, there it was an ideal called prime if ab is in I implies a is in I or b is in I.

Here, instead of asking for ab in I, we are saying ab is equal to 0; in other words, it is a statement about the 0 ideal. So, it is a simple exercise to verify that R is an integral domain if and only if the zero ideal is prime, this is a very simple exercise that I will not do, I will let you work it out because, if it is an integral domain you take the zero ideal if product of two elements in is in the zero ideal; that means, product of those elements is

0, because it is an integral domain one of them is 0. So, one of them must be in the zero ideal.

Similarly, if it is zero ideal is prime then, you can show that R is an integral domain. An equally simple exercise is that I so, let R be a ring and I is an ideal of R; at I be an ideal of R then we have the following equivalence.

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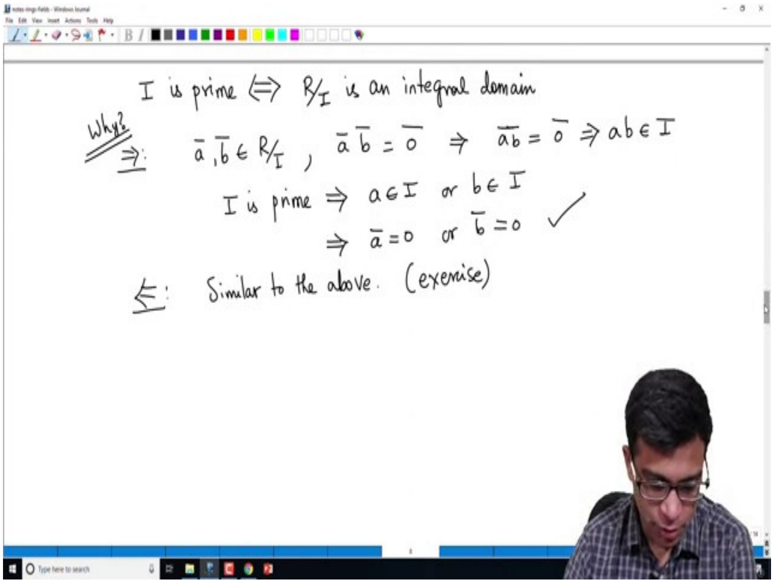

So, then we can say, I is prime if and only if R mod I is an integral domain. So, this is the characterization of prime ideals. In fact, it is sometimes useful to verify that ideals are prime using this condition. So, I is prime if and only if R mod I is an integral domain. So, this follows because of our understating of quotient rings and correspondence theorem of ideals and so on.

So, I will quickly tell you why this is true; why is this true so, why is this statement that if I is prime if and only if R mod I is an integral domain true. So, suppose I is prime; so, in this direction suppose I is prime. So, let us take two elements, what are two elements of R mod I, they are of the form R mod a bar comma b bar. So, suppose a bar times b bar is 0 bar right. So, remember in order to verify that ring is an integral domain we have to check that product of two things is zero implies, one of them is zero.

So, I am taking two things whose product is zero, but that means, remember a b bar; a bar b bar is just a b whole bar, this implies by the definition of a quotient ring if an element is 0 modulo I; that means, that element is in I. But since I is prime that is the assumption, if I is prime and a b is in I, a is in I or b is in I, but if a is in I or b is in I; that means, a bar is 0 or b bar is 0 which is exactly the statement for an integral domain.

So, if I is prime, R mod I is an integral domain; conversely if R mod I is an integral domain you want to show that I is prime. So, take two elements whose product is in I, but then take the residue so, I will leave that as an exercise similar to above ok. So, I leave it as an exercise, it is a good exercise for you to get used to the notion of quotient ring.

So, in other words to verify that an ideal is prime or not, it suffices to show that, verify that R mod I is an integral domain or not. So, now, some examples I will consider before moving on to maximal ideals.

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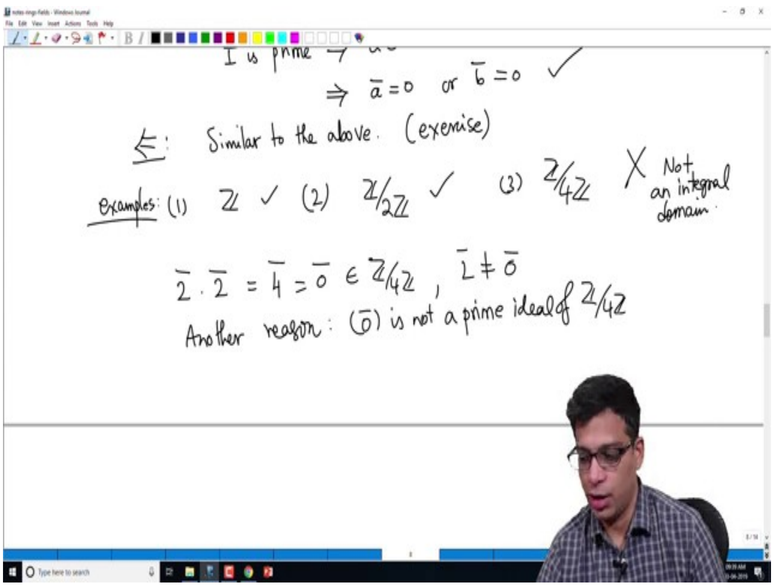

So, suppose R is an integral domain. So, as I told you earlier Z, Z is an integral domain; another example would be Z mod 2Z this is an integral domain this is an integral domain; what about Z mod 4Z, this is not an integral domain, let us see why; not an integral domain, why? So, why is this can you produce two elements in this ring whose product is zero, but neither element is 0, yes you can do that because 2 bar times 2 bar is 4 bar equals 0 bar in Z mod 4 Z, but 2 bar is not 0 bar ok.

So, this is another reason is that we already saw in the last video, 0 bar is not a prime ideal of 4 bar Z mod 4Z right; 0 bar is not a prime ideal hence Z mod 4Z is not a prime ideal by the observation I made here, integral domain if and only if 0 ideal is prime ideal.

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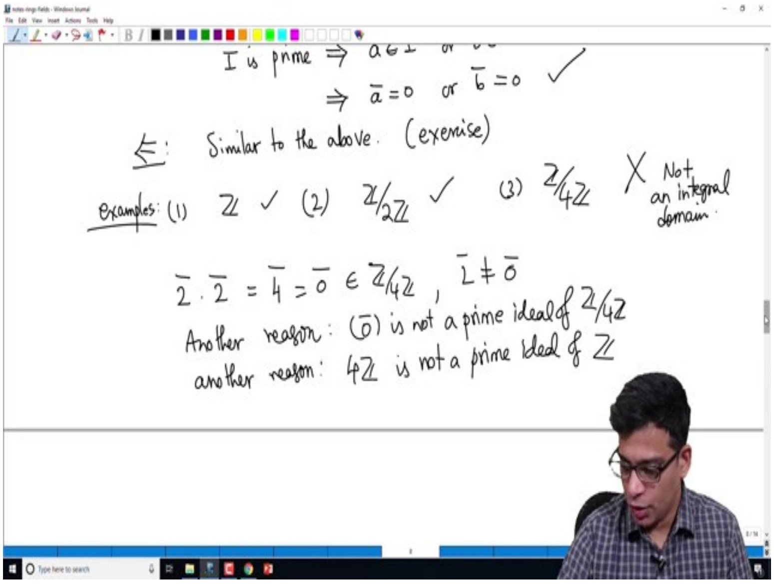

Yet another way to see this is the following 4Z sorry, I should not put brackets 4Z is not a prime ideal of Z that also we know because product of 2 with itself is in 4Z, but 2 is not in 4Z. So, 4Z is not a prime ideal of Z and hence by the exercise I left for you Z mod 4Z cannot be an integral domain. So, as you can see all these are connected, the notion of prime ideal and the notion of integral domain.

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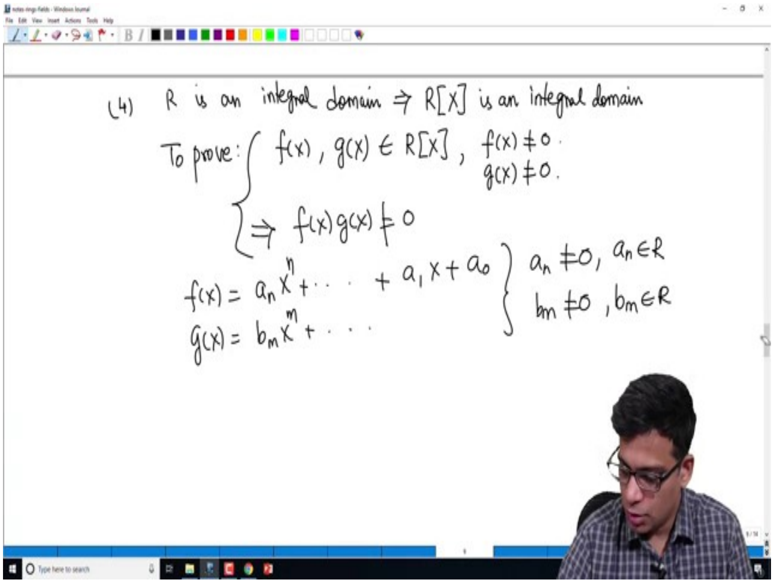

So, some other examples; so, fourth example if R is an integral domain; so, let us say R is an arbitrary integral domain then I claim that the polynomial ring in 1 variable. In fact, any number of variables over R is an integral domain. So, this requires a little bit of work, but it is not difficult at all, how do you show that R x is an integral domain? You want to show that any two polynomials if you take that are non zero. So, we want to prove that. So, if f x and g x are in R x, let say f x is non zero, g x is non zero, then f x times g x is non zero.

So, this is the contrapositive statement for definition of integral domain, you take two elements which are both non zero, their product is non zero, this is not difficult to show because. So, I will leave the details for you, if f x is a non zero polynomial write it as a n x n plus whatever I do not care what the remaining terms are, and similarly g x is b n x n, b m I will write and I do not care. And I know that a n is non zero; a n is an element of the ring of course, b m is non zero, b m is in R because f and g are non zero polynomials that leading coefficients will take to be non zero. We will take the largest coefficients so, the degree of f is n the degree of g is m.

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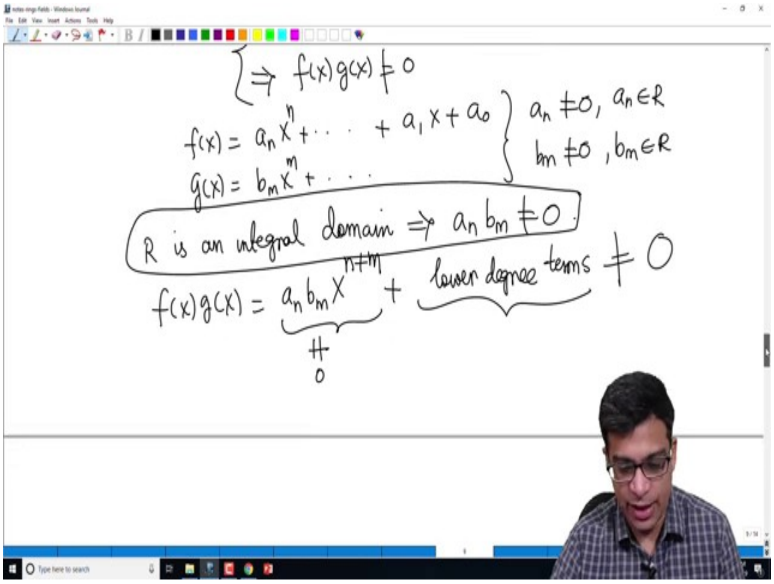

So, by definition, these are non zero, but because R is an integral domain R is an integral domain implies a n times b m is non zero, because a n and b m are non zero elements their product cannot be zero.

Now, what is f x times g x, the point is f x times g x, you will have to multiply two polynomials. So, they will be lots of terms, but there will be exactly one term with degree n plus m. So, these are lower degree terms, which I do not care about, right the largest power of x that you can find in the product of f and g is m plus n, the contribution coming from a n x n times b m x m.

So, when you take the product you will have this for every other term the degree will be strictly less than n plus m. So, they are lower degree terms; now because a n and b m are non zero elements in the ring; a n b m is non zero hence this is non zero and there can nothing else no subsequent term of the product can cancel this. So, this will survive and this is non zero ok. So, this proves that product of 2 non zero elements in the polynomial ring is non zero and the crucial statement is this: you need that a n b m is non zero. So, R is integral domain is of course, important.

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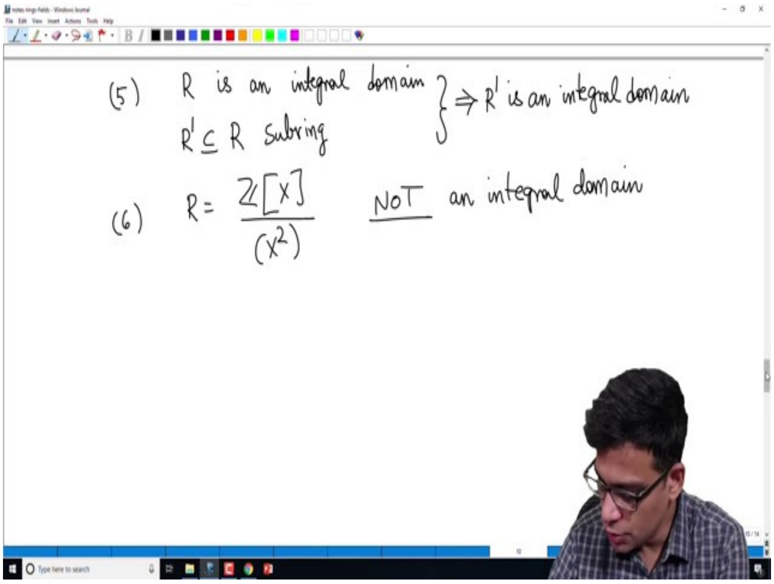

So, if not then R x cannot be an integral domain that is because R sits inside R x so, as an easy example. So, I will say that if R is an integral domain, another example which is related to my remark just now that I made; if R is an integral domain and let us say R prime is a subring of R. R prime is a subring of R; that means, remember that it is a subset which is closed under addition multiplication and it is in fact, in a subgroup under addition and so on. So, it is by itself as ring then R prime is also an integral domain; R prime is an integral domain that is clear right because if two elements you take from R prime, their product is 0 those two elements are in fact, in R and their product of course, is the same that is the nature of subring the product of R prime is same as product of R.

So, if two elements of R prime are multiplied you get 0, then their product in R is 0 because R is an integral domain one of them is zero. So, you conclude that R prime is an integral domain. So, this is easy; on the other hand if you take a integral domain and a bigger ring then you do not always get an integral domain. So, consider in this case R to be Z x modulo the ideal x squared. So, I claim that this is not an integral domain, why is this. So, let us check this.

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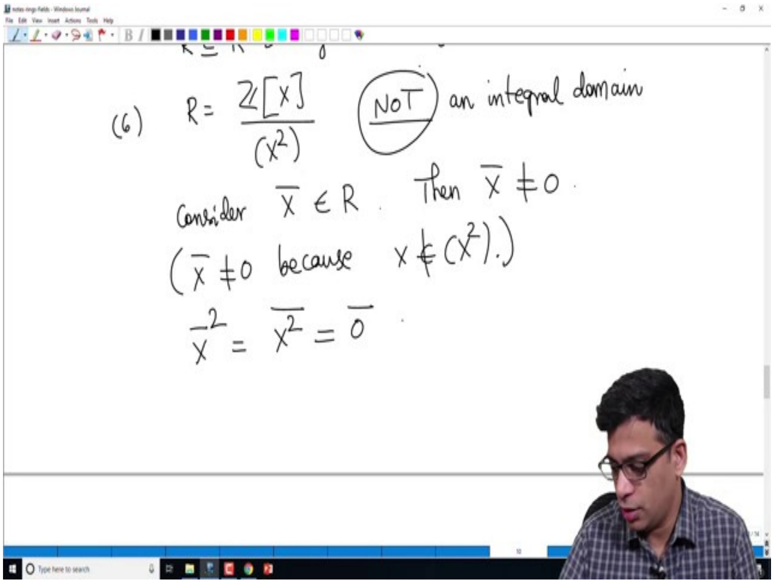

So, I claim that the ring quotient ring Z x by x squared is not an integral domain. So, in this case it is trivial because consider the residue of x. So, consider x bar and R, then x bar is non zero, why is this? x bar is non zero because x is not in the ideal x squared this is the reason. Remember elements residues are 0, if before taking the residue the element is in the ideal that you are quotienting.

x bar is 0 if and only if x is in the ideal that you are quotienting which is x squared, but certainly x is not in x squared because ideal generated by x squared is the collection of elements which are multiples of x squared. In particular if you take a nonzero element in the ideal x squared its degree will be at least two, whereas, x as degree one. So, x can't be in x squared. So, it is nonzero, but what about x bar squared; x bar squared is x squared bared, but x squared bar is 0 because x squared is in the ideal x squared.

So, you have a nonzero element x bar whose product with itself is 0 bar. So, our Z x mod x squared is not an integral domain, in other words the 0 ideal of this is not a prime ideal.

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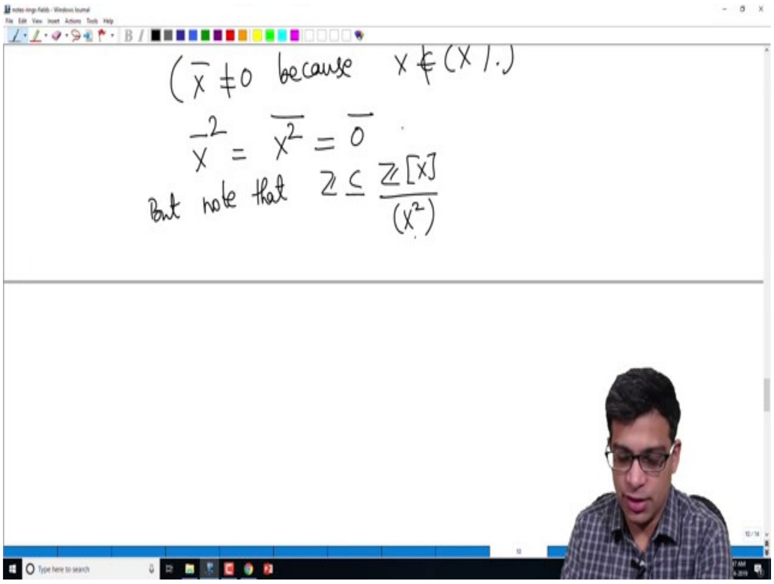

But remember Z x; Z can be thought of as sitting inside Z x mod x squared, because any integer is a polynomial right by it is a constant polynomial. So, you can take that as an element of this.

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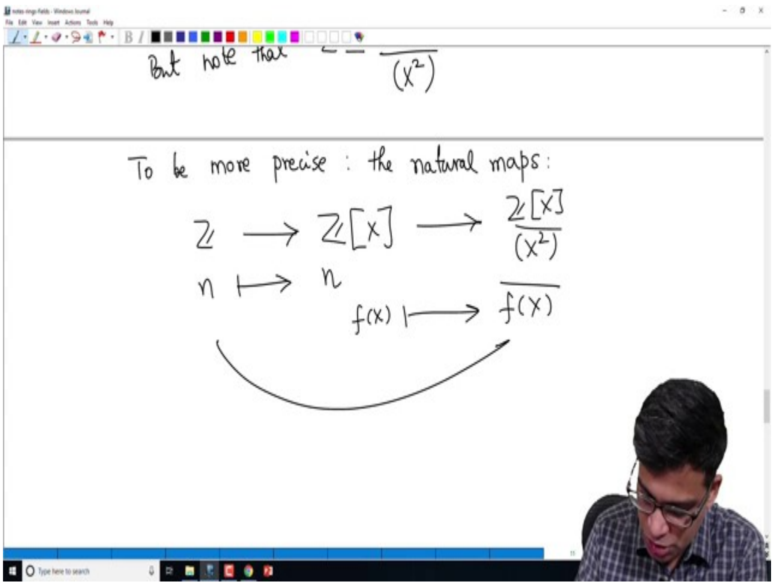

So, in other words to be more precise; to be more precise what I am saying is the natural map. So, we have natural maps, Z to Z x which is simply the inclusion of Z in so, an integer n goes to an integer n to we have also a natural map coming from the quotienting process. So, this here; so, here of course, f x goes to f x bar the second map. So, this is composition that you have is an is an injective map.

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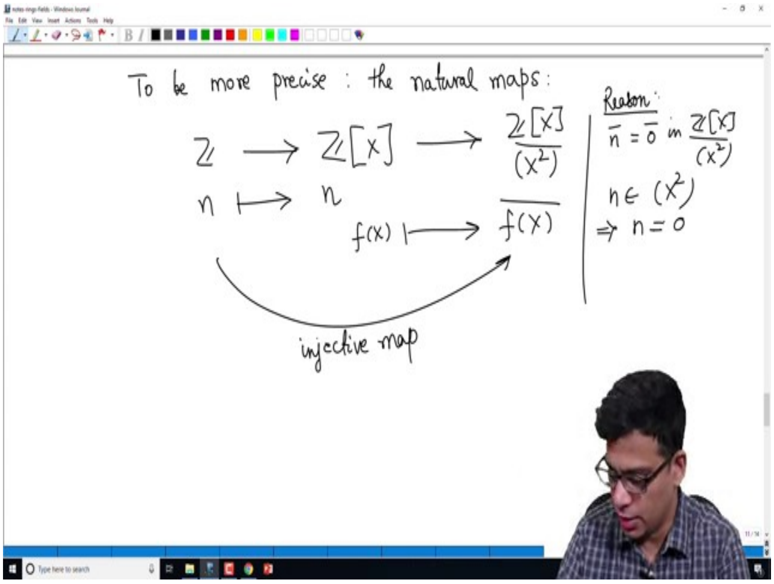

Remember that once you have any; so, this is an injective map because an integer is never inside; a nonzero integer is never inside the ideal x squared because if n goes to 0 if n bar. So, the reason is if n bar is 0 bar in Z x mod x squared; that means, n is in the ideal generate by x squared, but the only integer that is an ideal generate by x squared is 0.

So, the kernel is 0; that means, it is an injective map. Once it is an injective map Z is isomorphic its image by the first isomorphism theorem. So, you can think of Z as a subring of Z x mod x squared, Z is an integral domain but Z x mod x squared is not an integral domain. So, subrings of integral domains are integral domains, but if your bigger ring containing an integral domain is not necessarily an integral domain. So, these are some examples of integral domains, this notion of integral domain is intimately connected to the notion of prime ideals.

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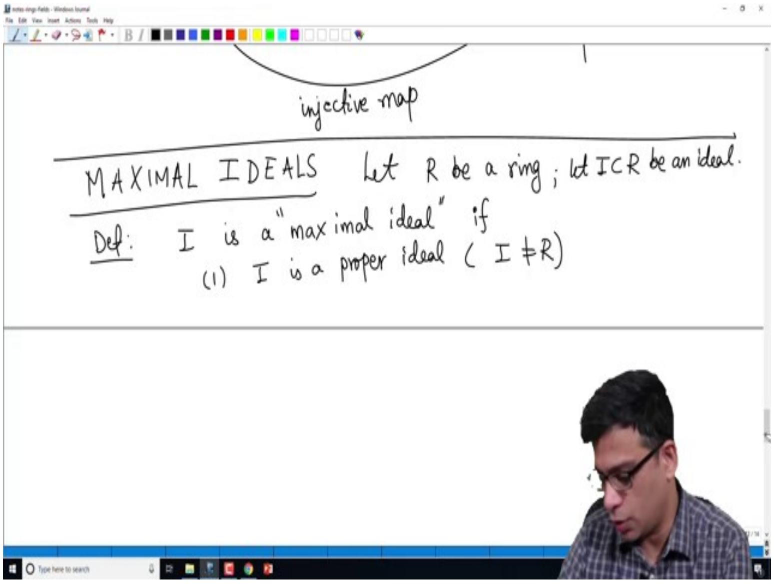

So, the next class of rings that I am going to study; so, ideals that I am going to study is the very important notion of maximal ideals. So, let me define these now; so, I am going to define maximal ideals. So, let R be any ring and let I be an ideal of R so, let R be a ring let I be an ideal in that ring R. We say that I is a "maximal ideal" or I is "maximal" if it is a maximal ideal so, there is no bigger ideal ok, but there is always one bigger ideal, if I is a proper ideal you can always consider R as an ideal that is bigger than I, but that is a trivial case. So, we do not want to consider that so, maximal except for the full ideal. So, I is a maximal ideal so, the one way to put this is I is maximal ideal if you have J is any ideal.

So, first of all we need two conditions: one is that I is a proper ideal. So, remember also max prime ideals are supposed to be proper, similarly maximal ideals are supposed to be proper; that means, I is not equal to R right. So, proper ideal means I is not equal to R.

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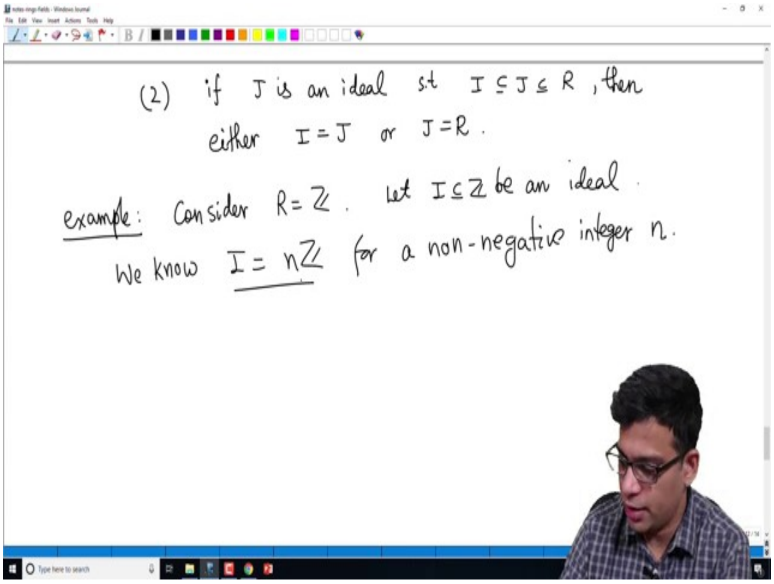

The second condition is I, if J is any ideal such that I is contained J which is contained in R of course, then either I is J or J is R ok. So, the only ideal that contains I; the only ideals that contain I are the ideal itself and the ring itself. So, these are the only two ideals that contain I otherwise it is maximal. So, there is nothing between nothing properly between I and R. So, that is a maximal ideal. So, I want to make some obvious comment here the first comment is. So, it is an example, consider the ring of integers, this our most familiar example right, for everything we start with the ring of integers.

So, what are maximal ideals; so, take any ideal, let I be an ideal. So, we know that any ideal of Z can be written as nZ for a non negative integer n. So, if you remember from the last video we started with this and determined when it is prime, now we are going to determine when it is maximal. So, let us take I equal to nZ and determine when under what conditions on n is I maximal.

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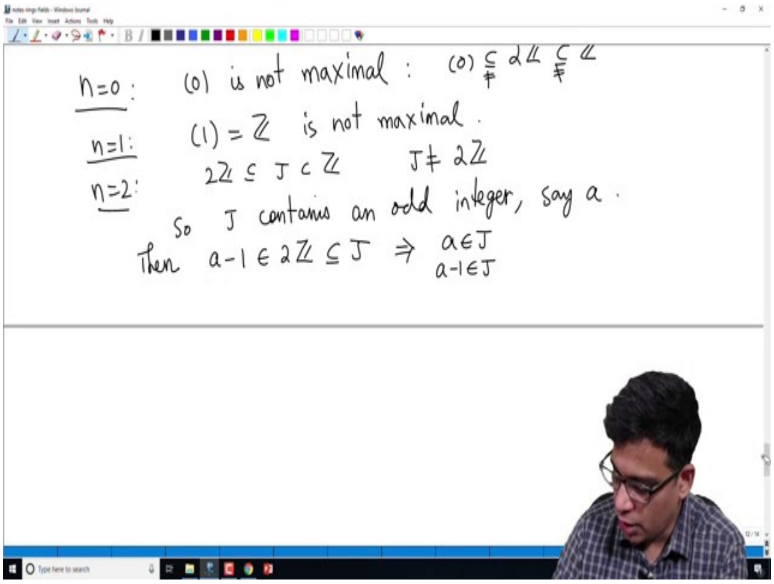

So, first consider n equal to 0, this is the obvious case first case. So, unlike the case of prime ideal where 0 ideal turned out to be prime here it is not maximal. This 0 ideal is clearly not maximal right, why is this? Because 0 ideal for example, is contained in 2Z contained in Z and 2Z is not equal to 0, 2Z is not equal to Z.

So, there is an ideal which is strictly bigger than 0 and which is strictly smaller than Z. So, it is not proper. So, it is not maximal now suppose on the other hand just some initial examples; n equal to 1 then we have the ideal generated by 1 which is Z of course, by definition this is not maximal, maximal ideals are proper ideals, what about n equal to 2 Z? Now this is interesting. So, let us see is 2Z maximal. So, let us suppose that 2Z is contained in J contained in Z, and suppose that J is not equal to 2 if it is equal to 2 then of course, we don't get any information. Suppose it is not equal to 2Z. Then; that means, J contains an odd integer. So, J contains an odd integer, say n, let us say a.

So, because J is not equal to 2Z, remember 2Z contains the set of all even integers, if J does not contain an odd integer, J would be equal to 2Z because J contains 2Z only way that it can be strictly bigger than 2Z is if it contains an odd integer let us call it a. But then we can write one for example, a minus 1 is in 2Z, because a is odd, a minus 1 is even, every even integer is in 2Z. So, a is in; but 2Z remember is contained in J this implies, a is in J and a minus 1 is in J, a is in J by hypothesis a minus 1 is in J because a minus 1 is an even integer.

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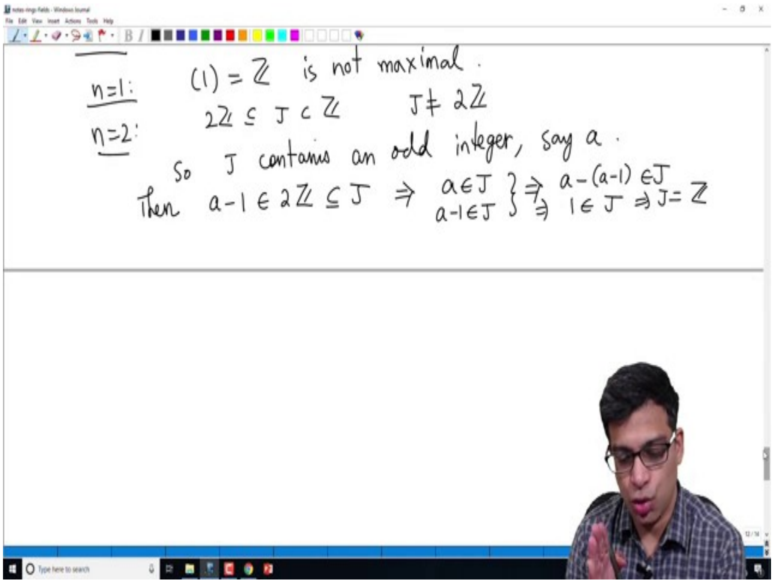

So, now if a minus 1 and a are both in J this means that a minus a minus 1 is in J; that means, 1 is in J; that means, J is Z right. If two consecutive integers are in J, the difference which is 1 is in J, but J is in then J is the unit ideal; now that proves that any ideal that contains 2Z properly is the full ring; that means 2Z is maximal right.

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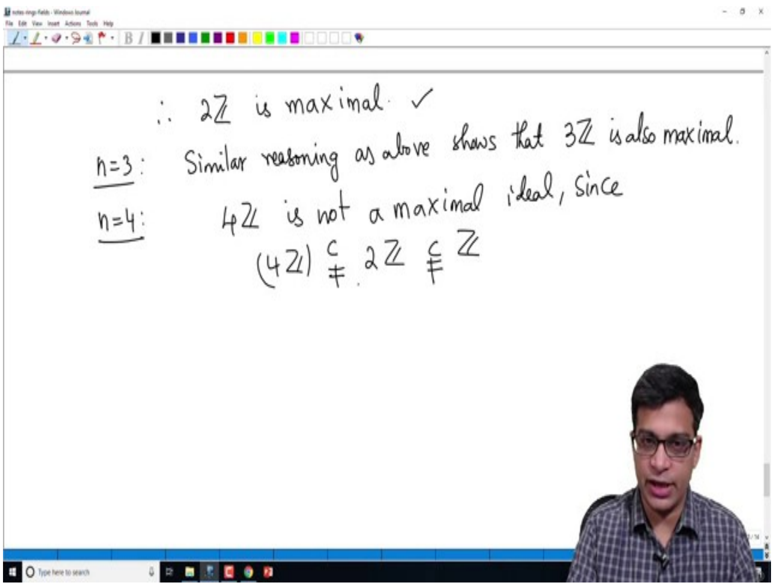

Because that is the definition of a maximal ideal, if you take any ideal that contains 2Z and that ideal is not 2Z we showed that that ideal is the full ring Z. So, it is a maximal ideal. What about n equal to 3, similar reasoning as above shows that 3Z is also maximal, if an ideal contains 3Z properly; that means, it contains an integer that is not a multiple of 3 by definition, using that and multiple of 3 you can write 1 as a linear combination. So, that is exactly the reason that we used in the case n equal to 2.

What about n equal to 4? 4Z is not maximal if you think about it for a minute, it is not a maximal ideal. Why is that? Since 4Z is contained in 2Z which is contained in Z this is proper, this is proper right. 2Z means all even integers, 4Z means all multiples of 4 certainly every multiple of 4 is even. So, 4Z is contained in 2Z, but 2Z contains 2, 4Z does not contain 2. So, this is a strictly bigger ideal and 2Z of course, is not the entire ring. So, there is something strictly in between 4Z and Z. So, this is a not a maximal ideal.

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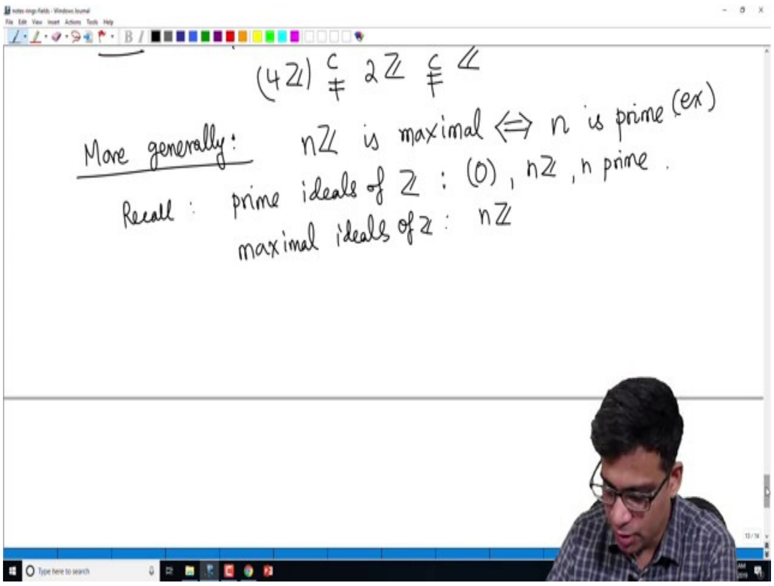

And more generally this is an exercise for you nZ is maximal if and only if n is prime ok. So, I will let you do this, I am going to do another proposition soon which will also prove this, but this you can directly show also nZ is maximal if and only if n is prime.

Remember in the examples that we have done, 2 and 3 happen to be maximal 4 is not maximal in that because 2 and 3 are prime 4 is not prime. So, nZ is maximal if and only if n is prime. So, if you compare prime ideals and maximal ideals in Z, recall prime ideals of Z are 0 and nZ where n is prime right, there was this additional prime ideal, 0 is a prime ideal of Z, what are maximal ideals of Z which we just computed by this exercise. So, this is an exercise for you, what are maximal ideals of Z? These are just nZ. So, every maximal ideal is in fact, a prime ideal, but there is a prime ideal that is not a maximal ideal. And so, that is a not an accident in general we have the following proposition.

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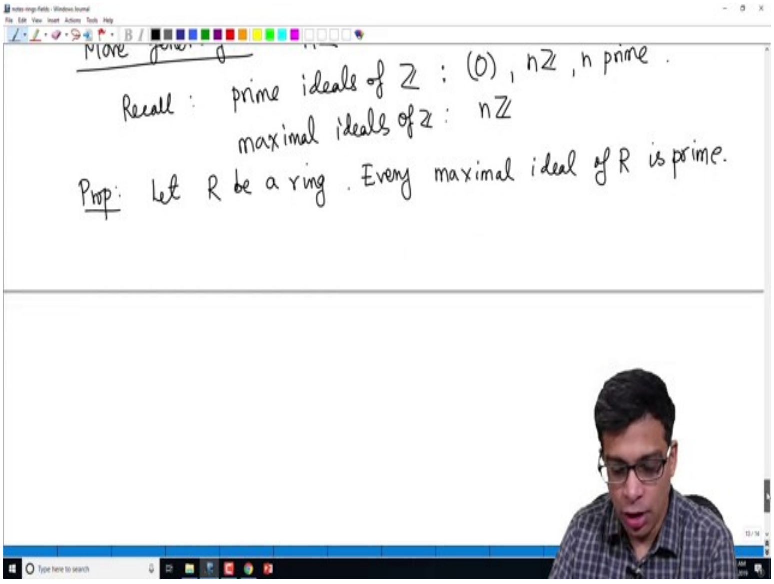

In any ring; let R be a ring every maximal ideal of R is prime. So, a maximal ideal is automatically prime, we know that the prime ideals are not maximal because the 0 ideal of the integers is prime because product of two integers is 0 means one of them is zero, but 0 ideal of Z is not maximal because the ideal 2Z contains 0Z properly and 2Z is a proper ideal.

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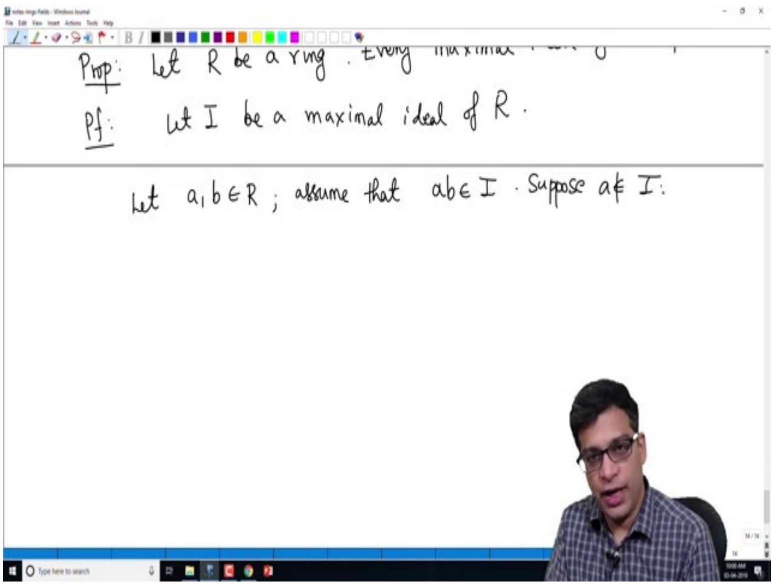

So, prime ideals are not always maximal; however, maximal ideals are always proper, so what is the proof? So, let I be a maximal ideal of R; so, let I be a maximal ideal of R so, I want to show that I is prime. So, definition of remember prime means if a product of two elements is an I 1 of them is an I.

So, let a comma b be elements of the ring R such that ab is in I, we will want to prove at the end of the prove that either a is in I or b is in I. So, suppose that if possible a is not in I. So, if a is in I we are done because we are trying to show one of them is in I, if a is in I we are done. If a is not in I we will show that in fact, b is in I then.

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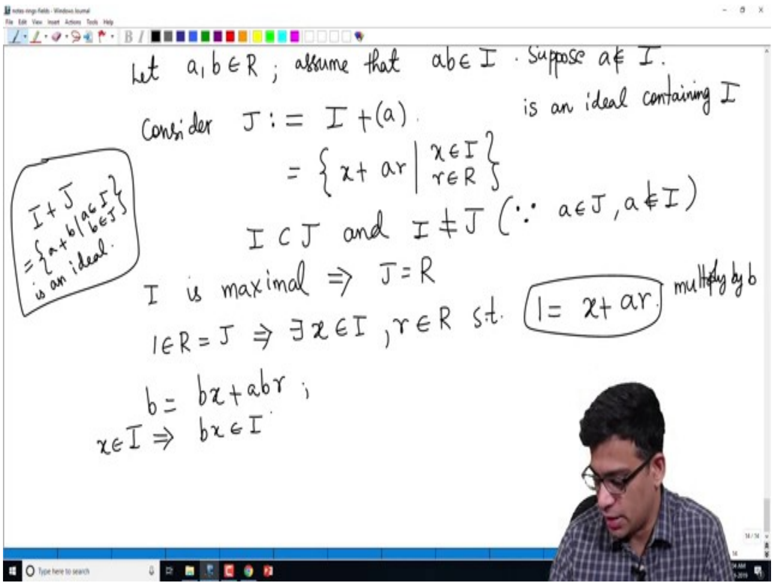

So, suppose a is not in I then consider the ideal J defined to be I plus a, if you remember I have defined ideal sums earlier in a previous video, but to recall that these are all elements of the form x plus ar, where x is in I and r is in R. This is a simple description of elements, but in general if I and J are two ideals I plus J is just r plus s or let us say a plus b, a is in I, b is in J, this turns out to be an ideal in fact.

If you remember this that is good otherwise please do this exercise if you have two ideals their sum is an ideal. So, in this case you take any element of I and any element of the ideal generated by a which is of the form ar because ideal generated by a is of the form elements of the form ar in R. Now, this clearly is an ideal containing I right because certainly you can take R to be 0, if x is in I, x plus 0 is in I plus a. So, x is in J; certainly I contains J and also I not equal to J this last point is because a is in J, but a is not in I by hypothesis. Remember a is in J because you can take x to be 0 and R to be one; that means, 0 plus a is in J

So, remember in general if we take two ideals and their sum, it is a bigger ideal it is an ideal that is generated by both of them. So, it contains both of them. So, an ideal J is the ideal generate by I and a. So, it contains a, but by hypothesis a is not in I. So, J is this ideal which is strictly bigger than I, but what do we know about I, I is maximal by hypothesis right, we have started with a maximal ideal; I is a maximal ideal which we are done. So, is prime; so, I is maximal means there cannot be any ideal between I and R properly. So, J is equal to R; J is properly bigger than I so, J must be the entire ring R; that means, 1 is in R always which is J; that means, 1 is in J. So, J can be written as something of this form. So, there exists x in I, r in R such that 1 is equal to x plus ar, right because every element of J is of the form x plus ar , where x is an element of I, r is an element of the ring we concluded that 1 is an element of R hence 1 is an element of J. So, there exists elements x and r such that x plus ar is equal to one.

But now let us look at this and multiply both sides by b; b is our other element. So, we get b is equal to b x plus a b r right b times 1 is b, b x plus b a r, this the distributivity of addition and multiplication. But x is in R sorry so, now, let us look at some obvious implications x is in R sorry, x is in I. In fact, right because x is in I; x is in I implies b x is in I; if x is in I, b x is in I.

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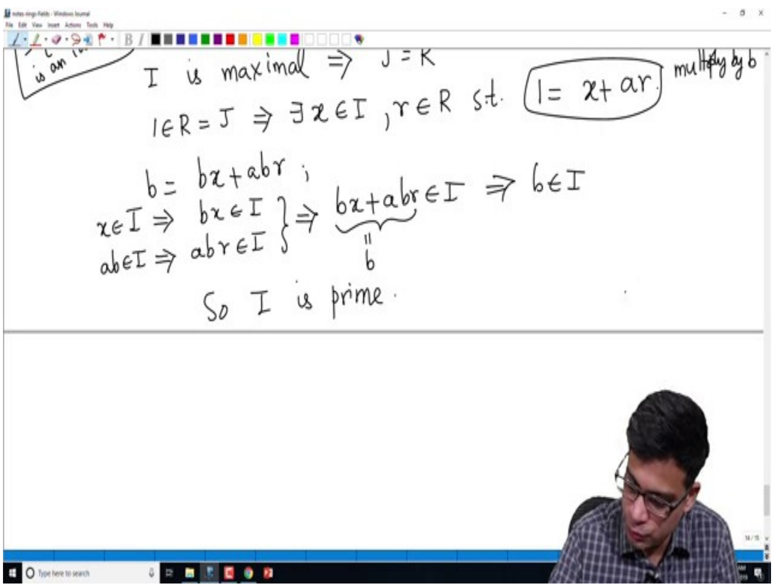

Correct because I is an ideal multiply anything by arbitrary ring element, we do not need anything about b here, in b is any ring element b x is in I and also by hypothesis ab is in I because we have taken two elements whose product is in I.

So, abr is in I. So, abx is in I, abr is in I though, hence their sum is in I right, bx is an element of I, abr is an element of I their sum is in I, but what is this sum? This is exactly b b x b x plus a b r this means b is in I. So, I is prime. So, what we have shown is if I is a maximal ideal and two elements in the ring have the property that their product is in I one of them must be in I so, I is prime. So, this completes the proof.

Now we go back to the exercise that I gave you where I asked you to show that an ideal nZ of the integers is maximal if and only if n is prime. This is now clear because if it is maximal ok so, one direction of it is clear at least if it is maximal then it is prime so, and it cannot be n cannot be 0. So, n must be prime; on the other hand, if n is prime you have to show that it is maximal. So, that part is an exercise for you ok.

So, let me stop this video here, in this video we looked at the notion of integral domains and we started talking about maximal ideals, we characterized maximal ideals of the integers and we also proved that any maximal ideal is a prime ideal.

Thank you.