Introduction To Rings And Fields
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Lecture - 14
Prime ideals

Let us continue the course now. So, far we have looked at various properties of rings, we looked at ideals in a ring and the most important thing we have so far looked at is the notion of quotient rings. If you have a ring R and an ideal we learnt how to construct a new ring using this pair R and I, called the quotient rings, quotient ring we denoted by R mod I. So, in today's video we are going to learn about two very special kinds of ideals.

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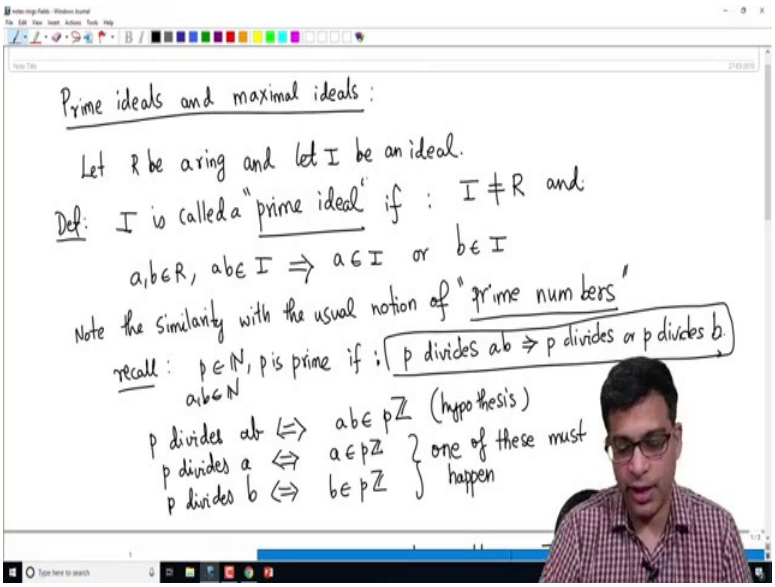

So, the goal today is to study prime ideals and maximal ideals. So, that is the goal today. So, these are special kinds of rings as I said. So, let us go ahead and define this. So, let R be a ring and let I be an ideal ok. So, remember that I is an additive subgroup of the ring R with the property that if you multiply an element of I with any element of the ring R, you will end again in the ideal I. So, that is what makes it an ideal ok.

So, now, I am going to define: I said to be prime, I is called a prime ideal if the following happens. If a and b are ring elements such that ab the product is an I ok. So, when I write a times b, I will just write a next to b, this implies either a is in I or b is in I ok. So, what

am I asking? I am asking for if you have a product of two ring elements is in the ideal, then one of the ring elements is in the ideal. So, this is supposed to give you, recall for you the notion of prime numbers. So, note the similarity with the usual notion of prime numbers that you learnt in school right. So, prime numbers ok. So, I will tell you why this is similar to that, but this is why we call this prime ideal ok.

So, the point is if you remember prime numbers, it says that p is a prime number if p divides, recall let us say p is a positive integer, p is prime if the following happens: p divides. So, let us a, b are also integers, p divides ab implies p divides a or p divides b ok. This may not be the way you have seen prime numbers, remember prime numbers are also defined as those that do not have any factors. So, 2 is a prime number, five is a prime number because they do not have any factors other than the obvious two factors 1 and that prime number. Whereas, 4 is a not a prime number because it has an additional factor, namely 2.

But that definition is equivalent to this is something that you may have seen before. So, this captures exactly what I am now generalizing to an arbitrary ring. So, using this we can conclude. So, remember p dividing ab means. So, I am I am trying to connect the notion of prime ideals that I have just defined to the notion of prime numbers that we learnt in school p dividing ab.

So, let me write it like this, formally write it like this. p divides ab if and only if this is our standard way of describing in English right. p divides ab, but in the ring theoretic language that we are learning in this course, this means that ab is in the ideal pZ. Remember p Z is the collection of all multiples of p. If p divides ab; that means, ab is a multiple of p so; that means, ab is in pZ ok.

So, now, what does it mean to say, p divides a? So, the same idea right p divides a if and only if a is in pZ similarly p divides b if and only if b divide b is in pZ so; that means, this is the exactly what you have seen here. If ab is in p Z this is the hypothesis, ab is in pZ the hypothesis is p divides ab that is equivalent ab being in pZ we want one of these must happen ok.

So, think of prime ideals as generalizations of the notion of prime numbers that we have seen in school ok.

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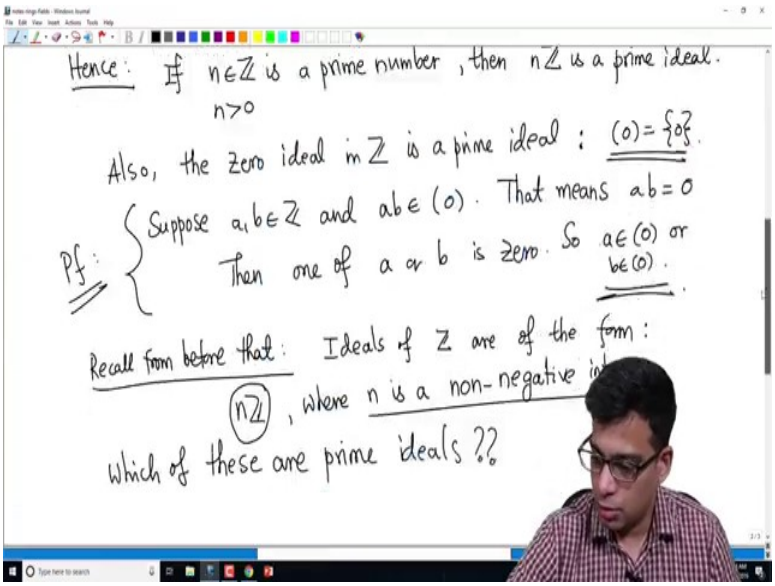

So, this connection makes it clear that. So, hence we have an ideal nZ in the ring of integers Z is prime implies n is prime ok. So, actually maybe I should not say this. So, let me say one direction. So, if n is a prime number n is an integer. So, let us assume n is a positive number n is a prime number because prime numbers are by definition positive integers then nZ is a prime ideal if n is a prime number then nZ is a prime ideal this is clear from the definition.

Because if n is a prime number this property holds; that means, if n divides a product n divides one of them, but that is because of this dictionary between division and belonging to ideals I have just described here. If nZ, if n is a prime number and a product belongs to nZ; that means, n divides that product because n is a prime number and divides one of them hence that element is in the ideal one nZ. So, by definition nZ is a prime ideal. I do not want to say the converse, but because I cannot say that if nZ is a prime ideal n is a prime number for the following reason; also the 0 ideal in Z is a prime ideal why is this? This is also an immediate consequence of the definition, remember 0 ideal consists of just the zero element, this is my notation for ideal.

I use this notation putting 0 in the bracket, but that is as a set just the set consisting of 0. But this is now how do you check that there is a prime ideal what you need to do is suppose a, b are integers and that ab belongs to the ideal 0; that means, ab equals 0 because there is only one element in the ideal generated by 0. So, ab is equal to 0, but if two inte-

gers give you two product of two integers is 0 then one of a or b must be 0 maybe both are 0, but I do not care, one of them is 0.

So, a is in the 0 ideal or b is in the 0 ideal ok. So, 0 is prime. So, this is the proof for the fact that 0 is a prime ideal. So, now, we learned in an earlier video that recall from before that ideals of Z are of the form. So, I proved this using Euclidean division algorithm that ideals of the ring of integers are of the form nZ, where n is a non negative integer, remember that 0 of course, is allowed 0Z gives you the 0 ideal. So, you have 0 ideal 0 and ideal 1Z which is the entire ring Z, you have ideal 2Z, 3Z, 4Z and so, on.

So, let us determine which of these are prime numbers? Which of these are prime ideals? So, we know from before that all ideals in the ring of integers are of the form nZ, where n can be any non negative integer; now the question is now that we have learnt what are prime ideals, which of these are prime ideals?

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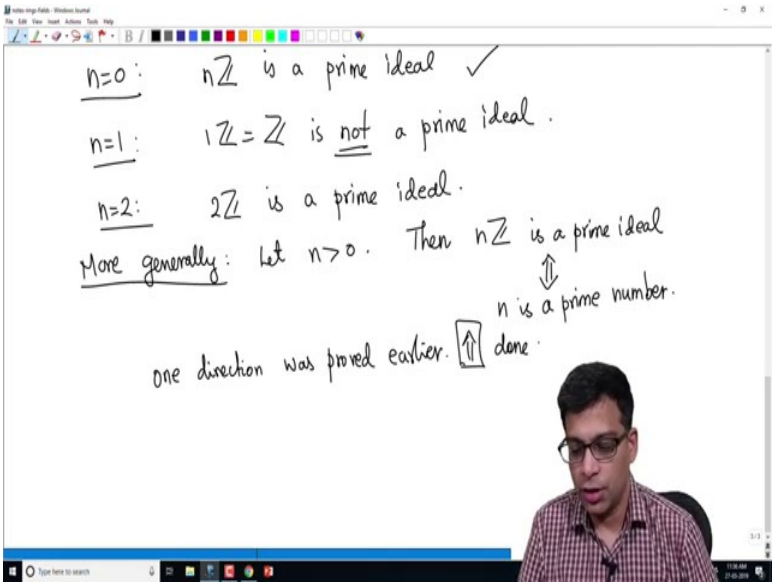

So, let us start with 0. So, when n is equal to 0 we have proved just now that nZ is a prime ideal right. So, this is ok. So, now, let us look at other integers. So, actually now I will remember I forgot an important part of the definition. So, let us go back here. So, let us go back to the definition of a prime ideal I is called a prime ideal if this happens, but I should add a very important condition that I forgot which is that I is not equal to R and this happens.

So, we do not want to call the full ring which is also an ideal of course a prime ideal. Prime ideals are by definition proper ideals. So, I is not equal to R by assumption and it has this condition ok. So, we will assume that we will remember that prime ideals are proper ideals. So, if you take one remember n equal to 1 you get 1Z which is actually Z all multiples of 1, that is all integers. So, this is not a prime ideal right this is not a prime ideal because I just added in the definition that the full ring is not a prime ideal ever. What about n equal to 2? 2Z is a prime ideal I claim ok.

So in fact, more generally let n be a positive integer, then nZ is a prime ideal if and only if n is a prime number, this is what I want to prove now. So, I claim that nZ is a prime ideal if and only if n is a prime number, remember every ideal is generated by nZ, n equal to 0 we have already taken care of, it is a prime ideal, n equal to 1 is the full ring. So, it is not a prime ideal. Now we are left with positive integers beginning with 2 and I claim that for each of those we can determine whether it is a prime ideal or not simply by looking at n. If n is a prime number its a prime ideal if n is not a prime number its not a prime ideal. So, why is this? One direction we already saw was proved earlier above right.

So, if n is a prime number if you check if you remember if n is a prime number then nZ is a prime ideal. So, I sort of said proved earlier if you go back and check the video. So, this is done. So, if n is a prime number we have shown that nZ is a prime ideal. So, that you can go back to the earlier part of the video and verify that you understand that proof now suppose we have the other direction.

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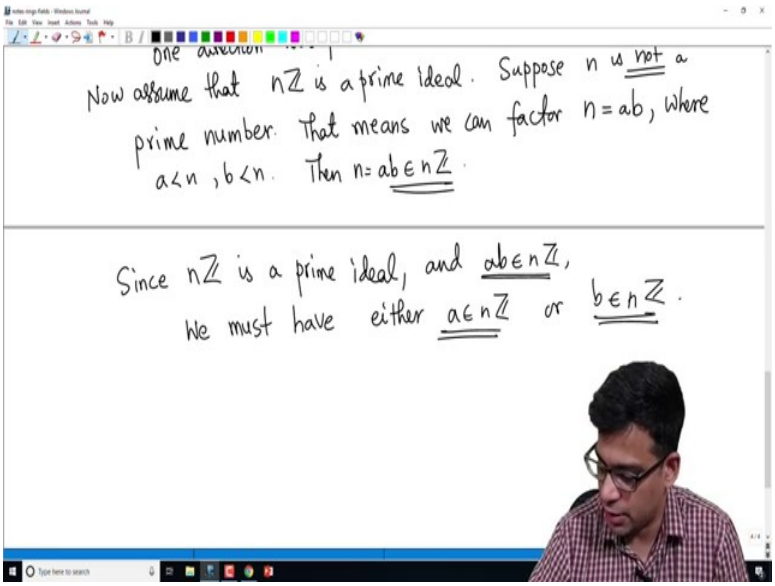

So, now assume that nZ is a prime ideal. So, assume that nZ is a prime ideal we want to show that n is a prime number. So, let us do that.

So, suppose n is not a prime number suppose n is not a prime number ok. So, now, we will use the alternative definition of prime numbers that you are familiar with school in school, not the definition I wrote earlier. What is an alternate definition? If n is not a prime number; that means, we can factor n as a times b where a and b are both less than n right, this is the definition of not being prime because it has a non trivial factor; that means, one and n are the only factors; that means, there is a factor called a so; obviously, that implies there is another factor called b such that ab is equal to n. So, this shows that n is not prime means n has such a product decomposition then remember ab is in nZ because, obviously, ab is equal to n.

So, nZ certainly contains n. So, ab is in nZ, but since n is nZ is prime is a prime ideal and ab is in nZ, we must have either a is in nZ or b is in nZ right this is the definition of a prime ideal that I gave in the beginning of this video. If you have a prime ideal in any ring; that means, if a product of two elements in the ring belongs to the ideal one of the elements must belong to the ideal. So, if ab is in nZ you have a is in nZ or here b is in nZ.

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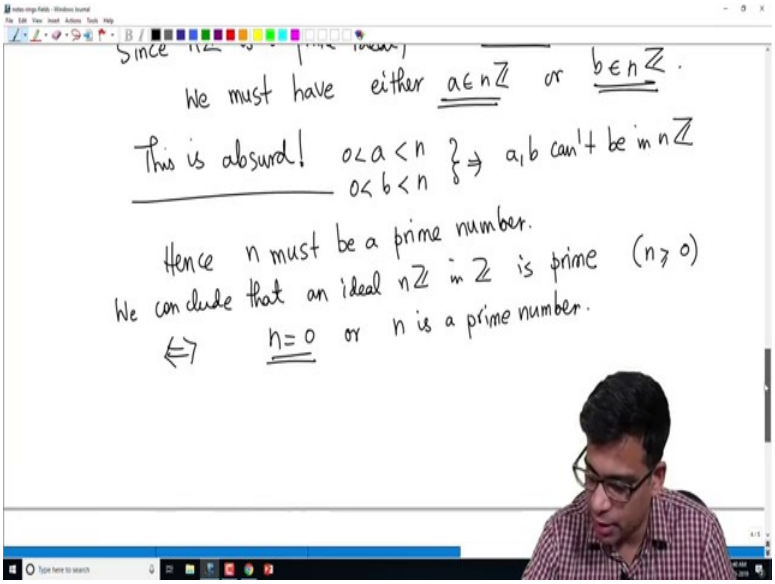

But now that is a problem, this is absurd, why is it absurd? This absurd because a is strictly less than n and a is of course, positive right.

Because n is positive and we have factored a into I should have said that earlier, but if n is the positive integer that is not a prime, we can factor n as a product of two positive integers less than n and less than n positive. So, this implies a, b cannot be in nZ because nZ is multiples of n. So, there is 0 and the next number in nZ is n, nothing between 0 and n can be in nZ so; that means, that we have a contradiction, hence n must be a prime number.

So, in conclusion what we have is, we conclude that an ideal nZ in Z is prime if and only if n is 0 or n is a prime number ok. So, assume n is positive of course, because every non-negative I should say every ideal in Z is written as nZ where n is a non negative integer. It is prime if and only if n is 0 or n is a prime integer prime number let us say ok.

So, this whole analysis is to give you the reasons for why we called these ideals prime ideals because they properly generalize the notion of prime numbers, but there is this additional ideal namely the 0 ideal which 0 is not called a prime number, but 0 ideal is a prime ideal ok. So, it is very much inspired by the notion of prime integers the notion of prime ideals.

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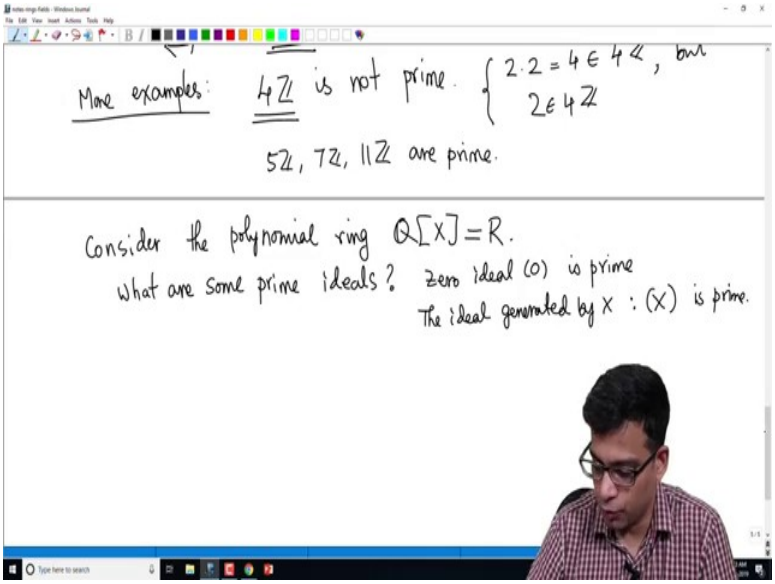

Now that we understand ideals, which ideals in Z are prime, let us look at more examples ok. Just to illustrate what we have just done we know that 4Z is not a prime ideal. So, from now on I am going to use this language, I will just say an ideal is prime or an ideal is not prime, I really mean that it is a prime ideal or not a prime ideal when I am talking about ideals I use this word like this. This is because we saw it in the proof, but I want to illustrate this again 2 times 2 which is 4 is in 4Z, but 2 is not in 4Z ok. So, this is exactly the proof that we gave earlier in the specific example this is what it looks like, 4Z is not prime. 5Z, 7Z, 11Z are prime and so, on ok.

So, let us look at more examples. So, now, look at other rings ok. So, consider the ring polynomial ring, let us say rational numbers are the coefficients and you have one variable. So, consider this as your R. So, in this case what are examples of prime ideals what are some prime ideals? So, unlike in the case of the ring of integers where we could completely classify all prime ideals, it is more difficult to list all the prime ideals in such rings, but we can certainly look at some examples of prime ideals. So, what are some prime ideals? 0 ideal is prime.

This is easy because the same idea, why is the 0 ideal in the ring of integers is prime. You take two polynomials whose product is 0, one of the polynomials must be 0. What about the ideal generated by X which we denote by (X). Remember our notation is you put an element in brackets; that means, it is the ideal generated by X, this is also prime.

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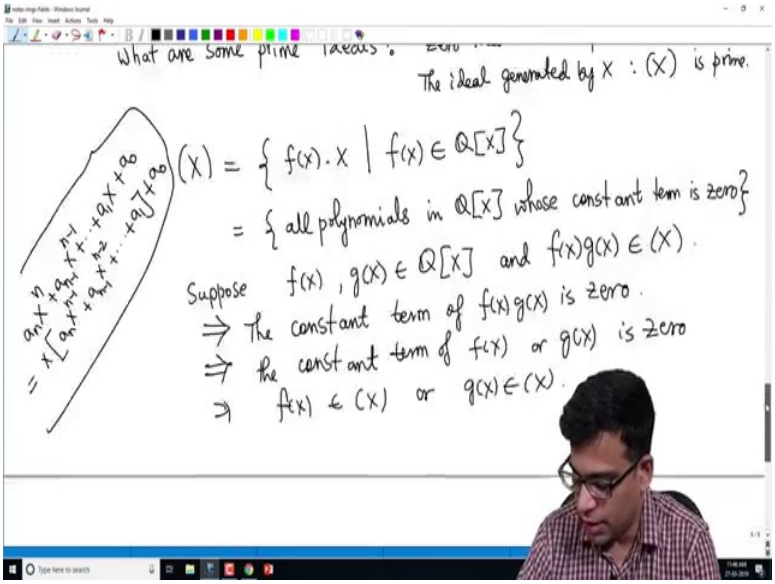

So let me prove this, first let us recall what is (X) the ideal generated by X, this is all polynomials which are multiples of X sorry its f X is in the polynomial ring, the ideal generated by X is all multiples of X, just like the ideal generated by 5 in the integers is all multiples of 5.

The ideal generated by X in Q X is all multiples of 5, but it is easy to describe this in a different way, these are all polynomials in Q X if it is a multiple of X, its constant term must be 0 and if the constant term is 0 it is a multiple of x because constant term. So, remember a polynomial looks like this an X n a n minus 1 X n minus 1 plus a 1 X plus a 0 right we can write this as X times an X n minus 1 plus an minus 1 X n minus 2 plus dot dot dot a 1, but you have to keep a 0 separately because a 0 has no factor of X. So, you cannot factor out X from that.

So, you have this, if it is a if a 0 is 0 it is a multiple of X if it is a multiple of X a 0 is 0. So, the constant term is 0 this is prime, one can check quickly because suppose f X and g X are polynomials in the polynomial ring Q X and suppose f X times g X is in the ideal generated by X; that means, the constant term of f X times g X is 0, but if you think in your mind about multiplying two polynomials f and g, and the product has constant term 0 then it is not difficult to see that the constant term of f or constant term of g must be 0 because the constant term of the product is simply the product of the constant terms.

So, the constant term. So, this implies the constant term of f X times g X is zero, but this means the constant term of f X or gX is 0; that means, f X is in the ideal generated by X or gX is in the ideal generated by X. If both have non-zero constant terms clearly the product will continue to have non-zero constant term because if you have product of two rational non-zero rational numbers is a rational non-zero rational number. So, X is prime.

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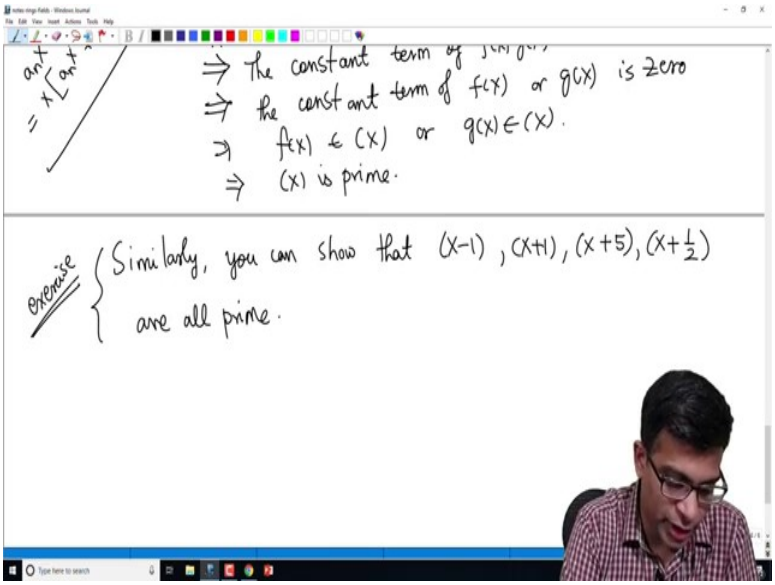

Using a similar argument you can show similarly you can show I will leave this as an exercise for you, for example, that X minus 1, X plus 1 X minus and so, on.

So, I will write one more example X plus 5, X plus 1 by 2 are all prime. Very similar to X, but now we are taking in this example we are taking the multiples of X minus 1, in this example we are taking the multiples of X plus 1 in this example we are taking the multiplies of X plus 5 and so, on. So, they are all prime, this is an exercise for you.

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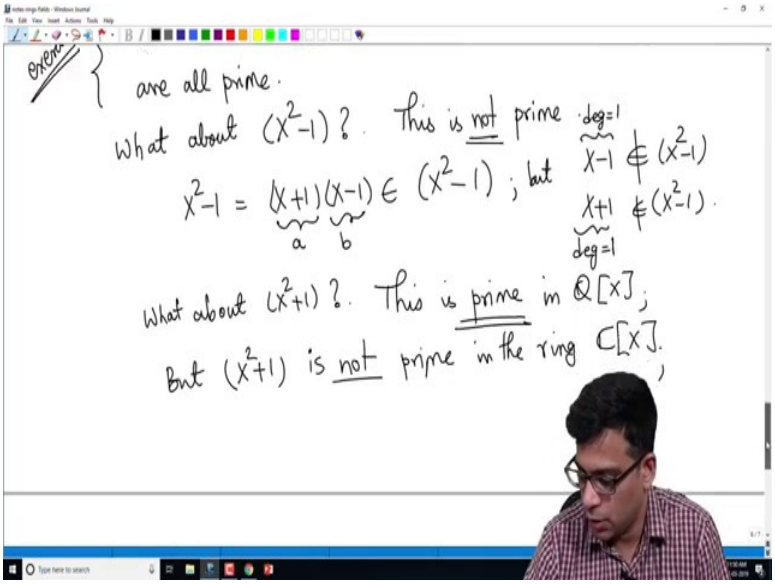

On the other hand let us look at another ideal, what about X squared minus 1? So, X squared minus 1 is the set of all multiples of X squared minus 1. So, this is not prime I claim, this is not a prime ideal why? Because if you take X squared minus 1 you can factor this as x plus 1 times x minus 1 this is in the ideal generated by x squared minus 1 ok.

So, but, so, this is a in my earlier notation this is in b. So, the product of a and b is in this, but X minus 1 cannot be in X squared minus 1, X plus 1 cannot be in X squared plus 1 X squared minus 1. This is for the simple reason that X squared minus 1 the ideal generated by X squared minus 1 contains all multiples of X squared minus 1. So, if you have any polynomial in it its degree must be more than 2 or the polynomial must be 0. The degree of this polynomial is 1 degree of this polynomial is 1. So, they cannot possibly be multiples of X squared minus 1. So, you have produced a product of two elements inside X squared minus 1, but neither of these two elements is in X squared minus 1. So, this is not a prime number prime I am sorry this is not a prime ideal ok.

So, this is where things get tricky. So, the reason that earlier examples were prime was you could not factor them roughly that is the reason. So, now, let us look at another one final example in the same ring Q X what about X squared plus 1 ok. This is prime I will not prove this for now I will prove this later, but the quick reason is that you cannot factor X squared minus 1 you cannot factor X squared plus 1 in the way that you could factor X squared minus 1 so, it becomes prime. So, let us to complete the circle of ideas I

should really say this is prime in Q X, but you can also consider this ideal in the polynomial ring over complex numbers.

So, X squared plus 1 is not prime in the ring C X because for exactly the same reason that we saw here that X squared minus 1 was not a prime.

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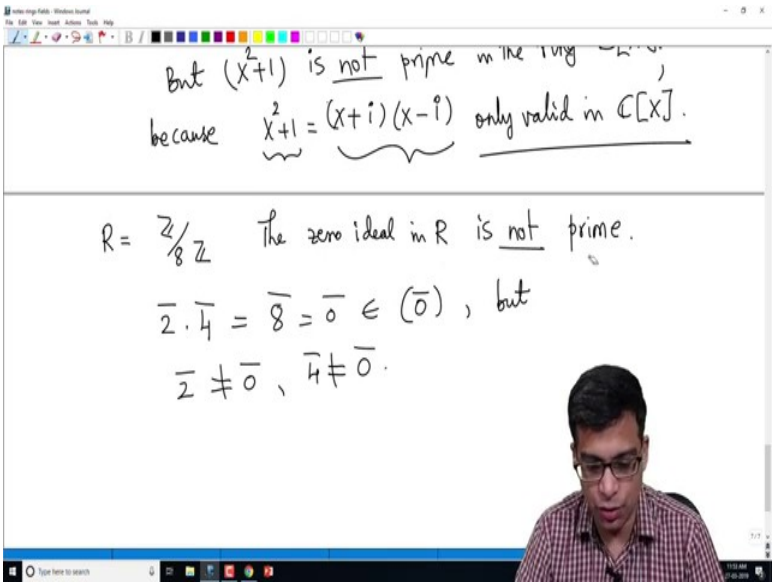

X squared plus 1 is not X squared plus 1 is not prime in C X because X squared plus 1 can be written as X plus i times X minus i. Remember this factorization is only valid in C X because X plus i is a polynomial in C X, it is not a polynomial in Q X or R X. So, you cannot factor this polynomial X squared plus 1 in Q X.

So, it is not a prime it is a prime ideal you can factor in C X and clearly for the same reason as before, X squared plus 1 is in the ideal generated by X ideal generated by X squared plus 1, but neither X plus i nor X minus i is in the ideal generated by X squared plus 1. So, this is not a prime ideal in C X. So, the same ideal if you change the coefficient ring from Q to R went from being prime to not prime.

So, this is a subtle point one have must keep in mind, you might be that find that an ideal is prime in one ring, but the ideal maybe considered as an ideal in another ring in which ring in which case it may not turn out to be prime ideal. So, this depends crucially on the ring the coefficient ring that you are considering. So, let me end the video with one final example given by Z mod 4Z.

Let me look at Z mod 8Z, let us take the ring Z mod 8Z this is the quotient ring remember that I defined in a previous video. So, consider the ideal 0 ideal the 0 ideal in R is not prime. So, I am giving this example because in all the examples have that we have looked at in this video, the 0 ideal is a prime ideal in Z it is a prime ideal, in Q X it is a prime ideal, but in this ring 0 ideal is not prime, why? That is because if you take 2 bar, remember I denote the elements of this ring with a bar on top 2 bar dot 4 bar is 8 bar which is 0 bar. This is in the ideal generated by z bar 0 bar, but 2 bar is not 0 bar right.

Similarly, 4 bar is not zero bar because 2 and 4 are non zero non zero modulo 8. So, they are not zero elements hence they do not belong to the 0 ideal yet their product is in the 0 ideal. So, the 0 ideal is not a prime ideal in the ring Z mod 8Z ok. So, there are a wide variety of examples, prime ideals are very important in ring theory, one needs to get used to this. So, hopefully you understood the definition and understood the various example in this video. So, I will stop the video now, in the next video we will continuous study of prime ideals and I will also introduce the notion maximal ideals.

Thank you.