Introduction To Rings And Fields

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Lecture - 10
Problems 3

OK, so let us continue with, this will be another video about problems. So, in the last video we were doing a problem about ideals in a ring and given two ideals I have talked about the intersection is an ideal, union is not in general an ideal, but we have defined a new ideal called I plus J which happens to be an ideal.

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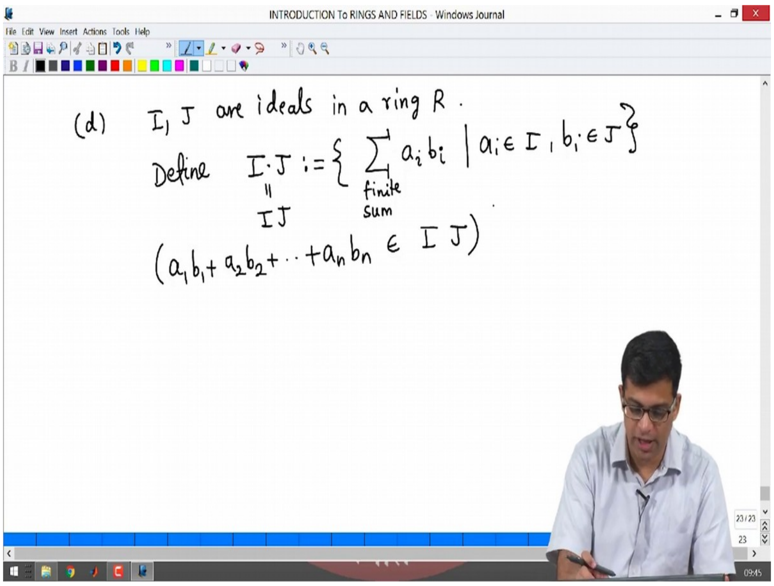

So, now I want to talk about the product of two ideals ok. So, I want to define IJ and it turns out to be the following. So, IJ is defined to be, so the setup is as before I and J are ideals, in a ring R, we will define I times J, which I usually just write IJ, is equal to I will take the sums of this form. So, a i b i, where a i is in I, b i is in J, ok. So, this is a finite sum. So, obviously, we want to take only finite sums because we can only add finitely many elements at a time.

So, what I am saying is for example, one, things are of this form. So, a 1 b 1 plus a 2 b 2, a n b n these are elements of IJ ok. So, just to give you explicitly what elements are. I take elements of I, I take elements of J, I multiply them and then I pool them together to get this. It turns out that this is important to define it this way. If you just take products of elements one single product, it is not an ideal. So now, I define it this way and I want to show that IJ is an ideal of R. I want to show IJ is an ideal of R.

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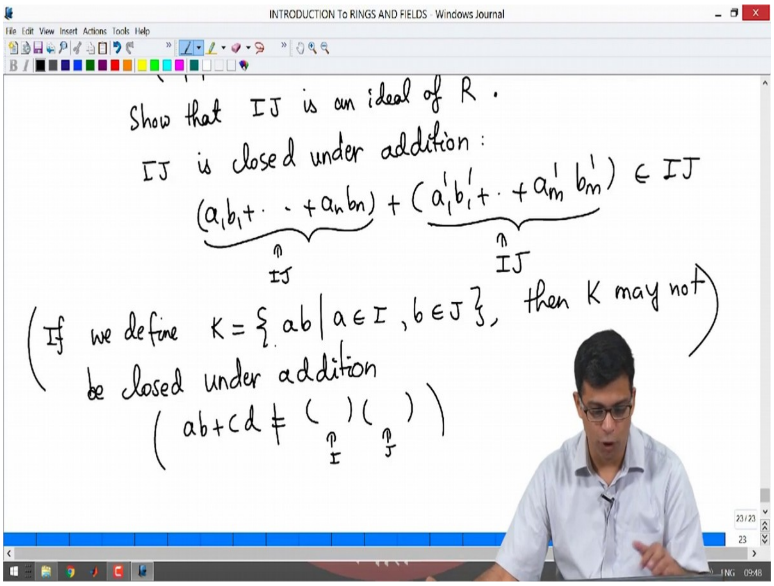

And this is a very simple actually. So, why is it an ideal? IJ is closed under addition because if you take two such things, so let us say a 1 b 1, a n b n, this is an IJ and let us take another one, so a 1 prime b 1 prime, a m prime b m prime, this is another element of IJ. There some obviously, will be in IJ because what is IJ? It is a finite sums of this form. So, you just put them all together and gets an IJ.

And this also tells you why if you just take a single term, if you take a b plus c d, so this, what I mean is if you define IJ to be not sums like this, but just a single term like this ab. So, if we define, let us say an ideal K to be a b such that a b a is an I b is in J then K may not be in general closed under addition, ok.

So, I will only make this remark. One can construct an example maybe later on I will talk about this. But if you play with rings a little bit you can find an example. If you just take one product it will not be closed under addition, because you take a b plus c d, you want to write this as something in a, something in I and another thing in J; but that may not be possible to do ok. So, it may not be like this. So, it is important in other words take not just a single a b, but a b plus a 1 b 1 plus a 2 b 2 plus a 3 b 3 and so on. Once you do that closed under addition is a trivial property. Is it closed under multiplication by ring element?

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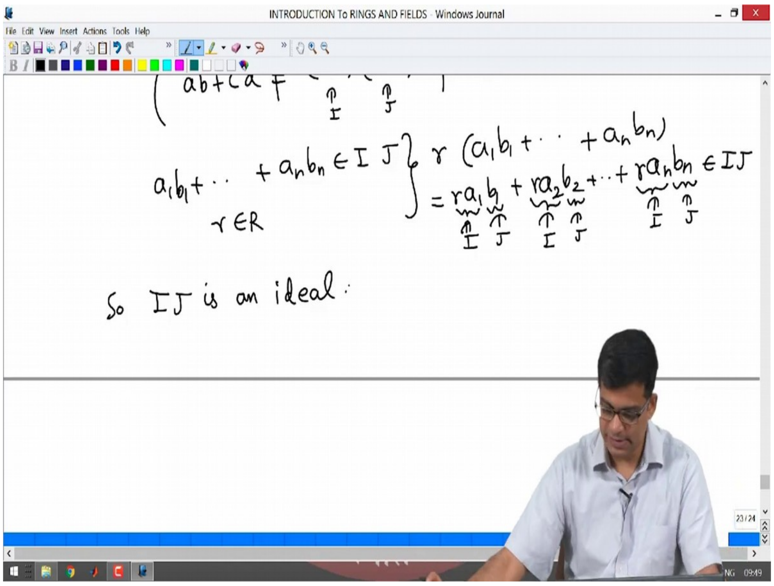

So, let us take an arbitrary element of J, IJ a 1 b 1, a n b n in IJ, r in R and let us take the product. So, r times a 1 b 1 plus a n b n is equal to r a 1 b 1, r a 2 b 2, r a n b n, right. So, if you do this the first one I claim is in you can think of this as this is in I, this is in J, right. This is in I, this is in J, this is in I, this is in J. So, you have each individual term is something in I time something in J, because a 1 is in I by hypothesis, r is a ring element, so r a 1 is in I, r a 2 is in I, r a n is in I. So, this is a sum of the correct kind. So, this is in IJ ok. So, we have, IJ is also closed under arbitrary ring multiplications. So, IJ is an ideal.

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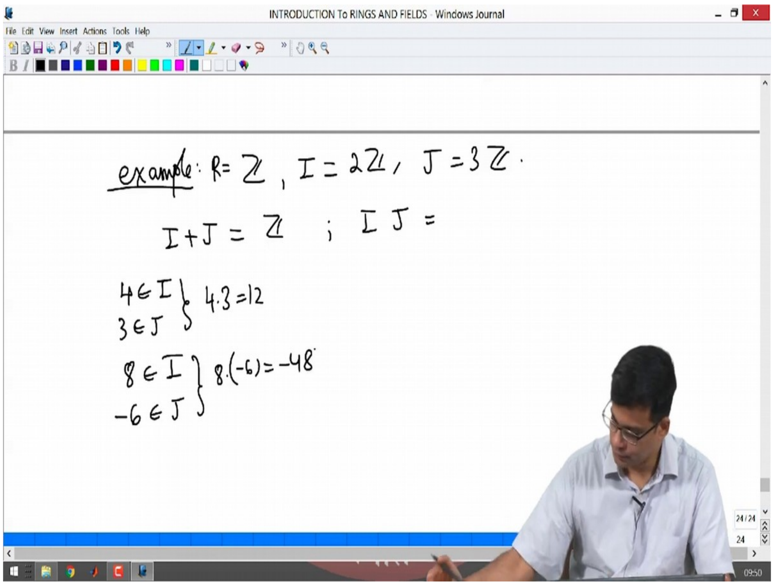

So, as an example, let us look at integers, R is integers, I as before 2Z, J as before 3Z. So, in this case we saw that I plus J is actually Z; I intersection. So now, let us compute I times J. I times J is you take something in 2Z, something in 3Z, you multiply them. So, for example, let us take 4 and 3 ok. So, 4 is in I, 3 is in J, 4 times 3 which is 12. So, you take that 12 and you can take another element. So, let us say 8 is in J, minus 6 is in sorry 8 is in I, minus 6 is in J. So, 8 times minus 6 is minus 48.

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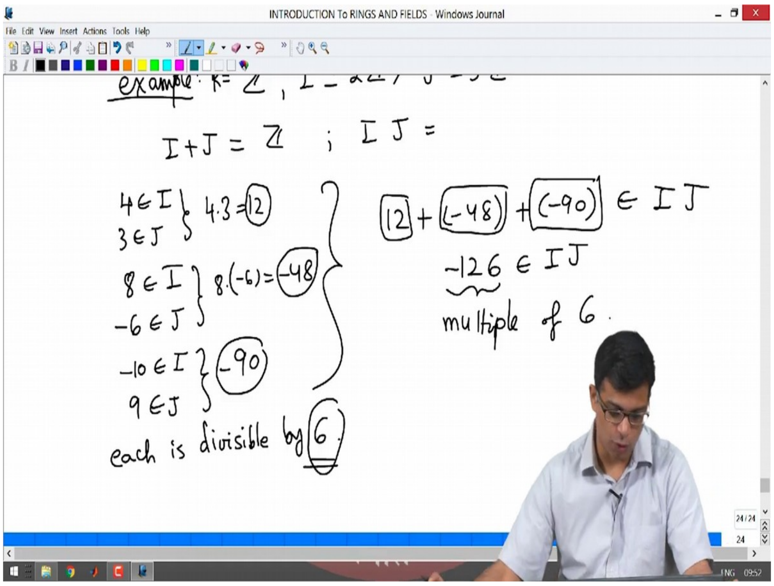

So, I will do one more thing minus 10 is in I, 9 is in J, so minus 90 is in, minus 90 is the product. So, you take the sum of these 3 things. So, 12 plus minus 48 plus minus 90 is in IJ, right. And what is this? So, this is minus; so, 90 plus 48 is 138, 126, right. So, this is minus 126, ok. So, that is one element of I J. So, you can construct lots of such elements.

Now, how do you describe this? It seems wild right. You can construct lots of elements like this. But one idea will make it clear to you. If you think about this each individual term here, I get 12, I get minus 48, I get minus 90, each of them is divisible by 6. That is easy if you think about this, because something coming from 2Z, another thing coming from 3Z. So that means, something is a multiple of 2, another thing is a multiple of 3, their product is a multiple of 6. So, everything is a multiple of 6. So, this is a multiple of 6, this is a multiple of 6, this is a multiple of 6, obviously, the sum is a multiple of 6.

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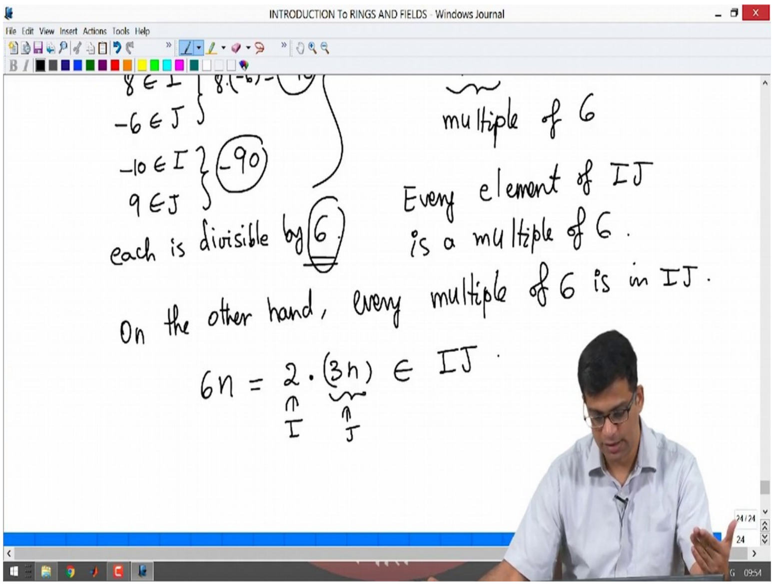

So, what I have just shown is that every element of IJ is a multiple of 6, you agree, ok. So, think about this, it is not difficult at all. Every element of IJ is a sum of products, each such product comes from something in I times something in J. Something in I is a multiple of 2, something in J is a multiple of 3, so their product is a multiple of 6. So, you have a sum of some finitely many multiples of 6. So, their sum is also a multiple of 6.

So, IJ contains only multiples of 6s, multiples of 6, every element of IJ is a multiple of 6. On the other hand, every multiple of 6 is in IJ. This is actually much easier, every multiple of 6 is in IJ. Why is that? What is a multiple of 6? Multiple of 6 is of the form 6 times n, right, it is something times 6. 6 times n can be written as, for example, 2 times 3 n, right, 6 n is 2 times 3 n. So, this is in I by definition, this is in J. So, 6 n is in IJ. In fact, 6 n is just a single product, you do not even have to take finite sum of such products.

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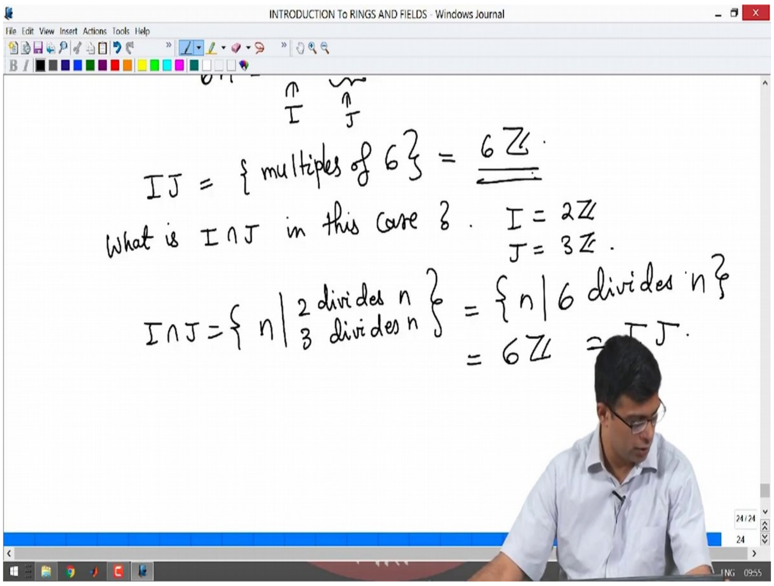

So, IJ contains all multiples of 6, every element of IJ is a multiple of 6, so I J is exactly multiples of 6. And there is a symbol for that right, IJ is precisely 6Z. So, I will write that here, IJ is 6Z ok. Just to finish this problem, let us also compute I intersection J in this case. What is I intersection J in this case? So, again remember I is 2Z, J is 3Z. So, what is I intersection J? So, I intersection J is all integers n such that n is in I that means, 2 divides n; n is in 3Z, so 3 divides n.

So, this is all integers which are divisible by 2 and divisible by 3. But then this exactly means 6 divides n, so 6 divides n, but that is exactly the set 6Z ok. So, in this case I Z is equal to IJ, I intersection J is equal to IJ. So, in this specific example we computed I plus J. So, I plus J is often the biggest ideal that you can construct from I and J. IJ and I intersection J are inside I; they are smaller ideals. Sometimes they happen to be equal, ok. So, these are some operations on ideals.

So, now, let us continue. What I want to do next is continue talking about ideals. So, let us say, going back to see the numbering now. So, we have done 6 problems, let us continue with 7th problem ok.

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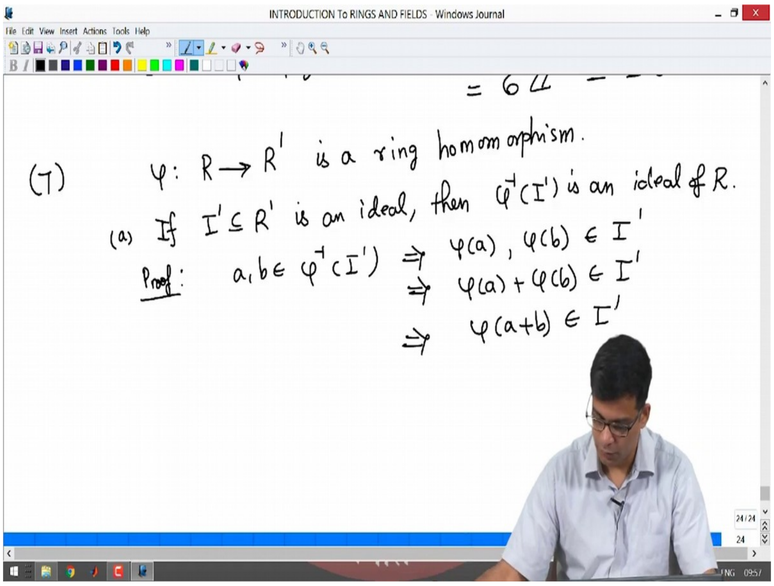

So, 7th problem is also about ideals. Let us look at the following. So, let us say phi from R to R prime is a ring homomorphism ok. So, what I want to do is, I want to talk about images and inverse images of ideals. So, first point is if I prime is an ideal of R prime, then phi inverse I prime is an ideal of R ok. So, this is fairly straight forward to show ok. So, this is the first point.

You take the inverse image of an ideal, it is an ideal. So, why is this? Proof. So, again, it is very clear. So, all you have to do is its closed under addition, it is closed under multiplication by an arbitrary ring element. So, let us say a and b are in phi inverse I prime ok, that means, phi a and phi b are in I prime, right, by definition because a b are in the inverse image means their images are in I prime. But I prime is an ideal, so phi a plus phi b is in I prime, but phi a plus phi b is phi a plus b by the definition of a ring homomorphism. So, this is in I prime.

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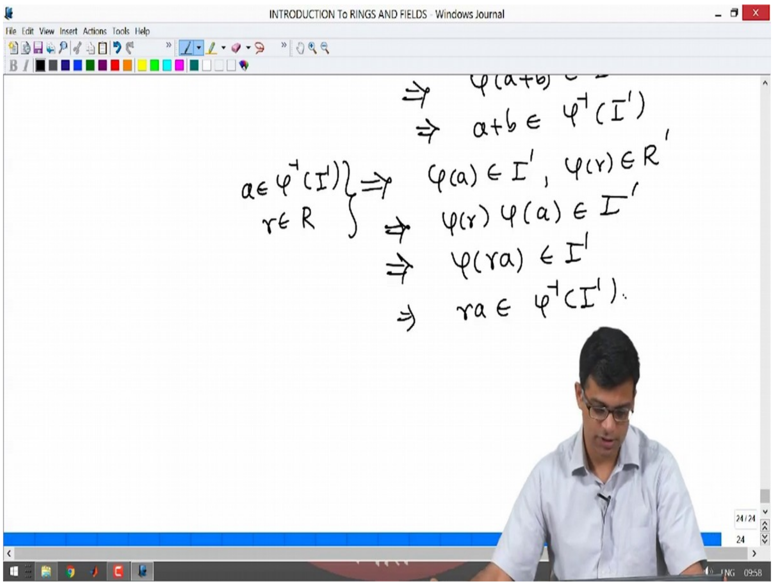

But phi a plus b is an I prime a plus b is in phi inverse of I prime, right. So, we started with a two arbitrary elements of phi inverse I prime, we have shown that their sum is in phi inverse I prime.

Now, let us take a in phi inverse I prime and r in R ok. So now, this data implies phi a is in I prime, phi r is in R, R prime, right. r is an arbitrary element of capital R, so its image is in capital R prime; a is an element of phi inverse a prime, so phi a is an I prime. So, their product because I prime is a ring a ideal, their product is an I prime so that means, phi of r a is an I prime because phi is a homomorphism that means, r a is in phi inverse I prime. So, this is very straightforward. Inverse image of an ideal is an ideal.

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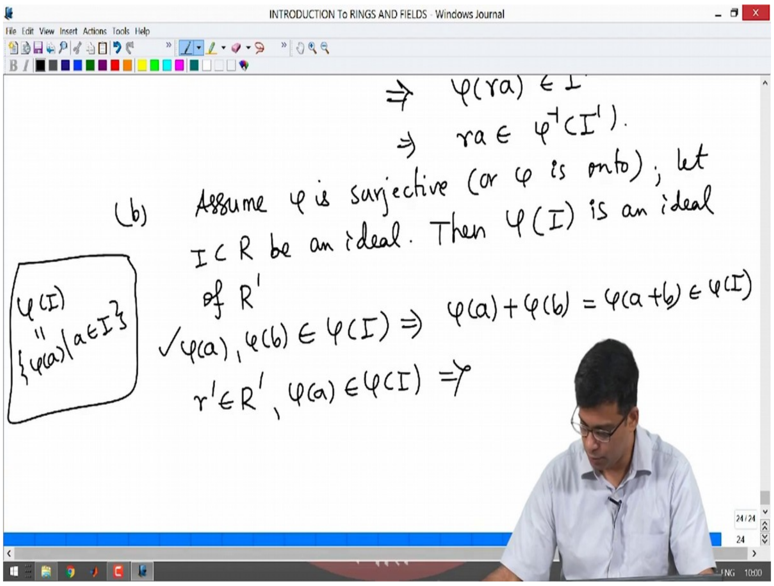

What about the image of an ideal? Here we will have to assume that phi surjective. What is surjective mean? Or another word for it is phi is onto. That means, every element of R prime is image of something. So, the image is equal to R prime. Let I be an ideal, then phi of I is an ideal of R prime. Phi of, image of an ideal is an ideal, provided the map is surjective. What is their argument? Very simple again, so let us take two elements of phi of I. What are two elements of phi of I? They are of the form phi a, phi b, right.

The images precisely, images of elements, images image of an ideal or image of any set is the set of all images of that set, of elements of that set. So, let us take two elements, phi a plus phi b is phi a plus b, by definition of homomorphism. This is of course in I, right because it is of course in phi of I because it is the image of a plus b; phi of I is the set of elements of the form phi a in I ok. So, this is ok.

Now, let us take an arbitrary ring element of R prime and phi of a in phi of I. And here is where we need the surjectivity of phi, because if r prime is in R prime, small r prime is in R prime, there is no reason to think assume that it is the image of something, but because phi is, since phi is onto there exists r in R such that r prime is phi r.

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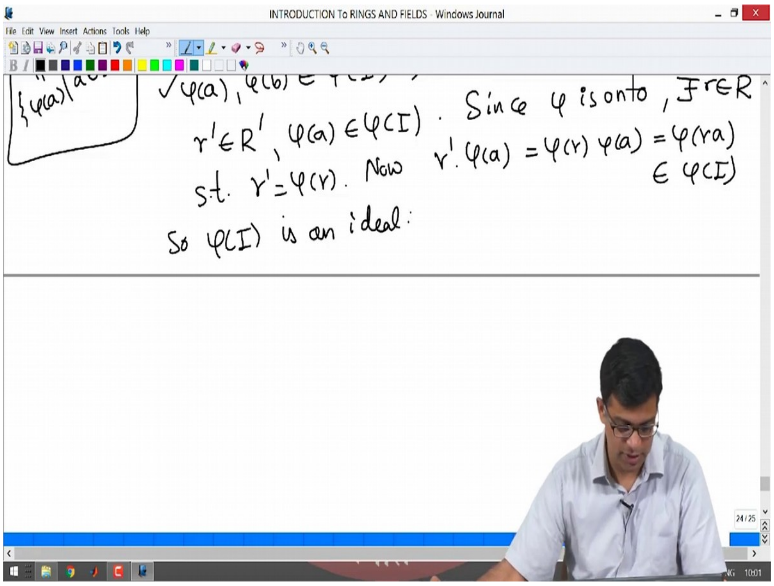

Now, r prime times phi of a is phi r times phi of a right, because r prime is phi r. This is phi of r a and this is in I, phi of I. So, phi of I is an ideal.

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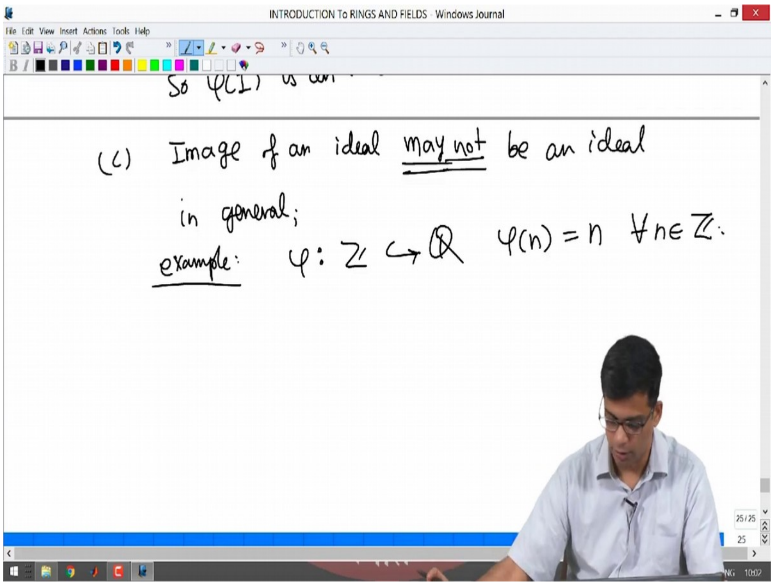

So, the last exercise here is, in this exercise last part is image of an ideal may not be an ideal in general ok. So, for example; so we need in other words we need the onto property. So, for example, consider; for example, consider the map from Z to R, to rational numbers. So, here the map is an inclusion. So, I simply take, so an integer and I map it to that integer itself. So, I claim that here image of an ideal is not an ideal.

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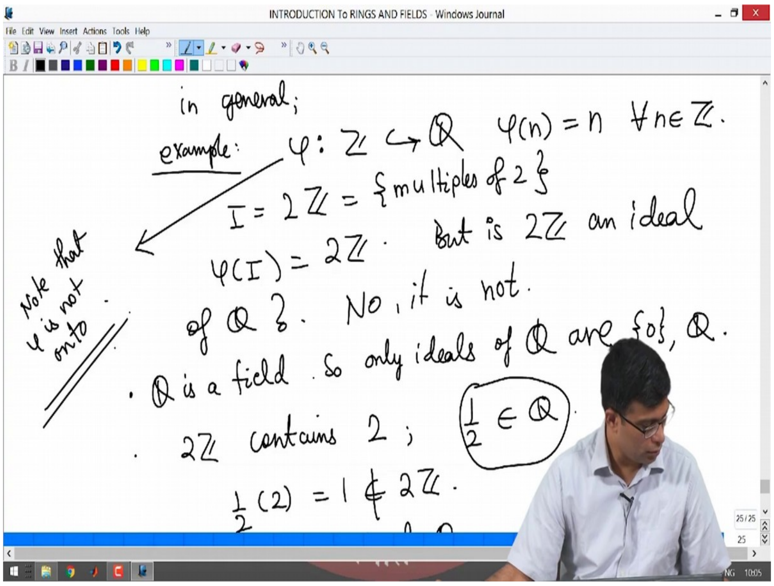

For example, take I to be 2Z the ideal 2Z that we have considered earlier. What is the image of 2Z? It is just 2Z, right. These are all multiples of 2 and the map phi is actually just the inclusion map. So, you take an integer and you put it inside rational numbers as it is so, image is 2Z. But, is 2Z an ideal of Q? So, the image is just 2Z, so you take the set of even integers in other words the set of multiples of 2, is that an ideal of Q? No, it is not. Why not? For one thing, Q is a field, we know that the only ideals of Q, so the only ideals of Q are 0 and Q.

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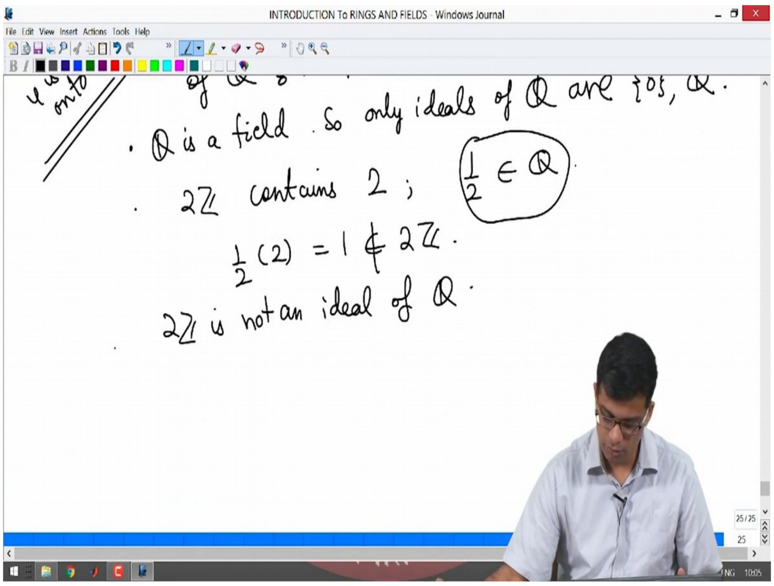

By an earlier problem we know that the only ideals of Q are 0 and Q, so Q, 2Z is not an ideal. So, also we know that more directly, 2Z contains 2, 1 by 2 is in Q right, 1 by 2 is an element of the rational numbers, but 1 by 2 times 2 which is 1 is not in 2Z right, 1 is not an even integer. So, 1 is not in 2Z. So, why is this not a problem in the integers? Why is 2Z an ideal in integers? That is because 1 by 2 is not available in Z to do this example ok. So, 1 by 2 is available in Q, so 1 by 2 multiplied by anything in 2Z should be in 2Z, for 2Z to be an ideal, but it is not. So, 2Z is not ideal of Q.

So, image of an ideal need not be an ideal if the map is not onto, because here of course, the map is not onto, right. The function from Z to Q is not an onto function, there are lots of rational numbers which are not images of this map. So, note, that of course, if phi is onto we have shown in the earlier part that image of an ideal is an ideal, but if it is not onto it need not be as this example shows ok. So now, I will do one more example and then we will stop this video and continue with other things in the next video ok. So, let us see what is the number now? So, number 8.

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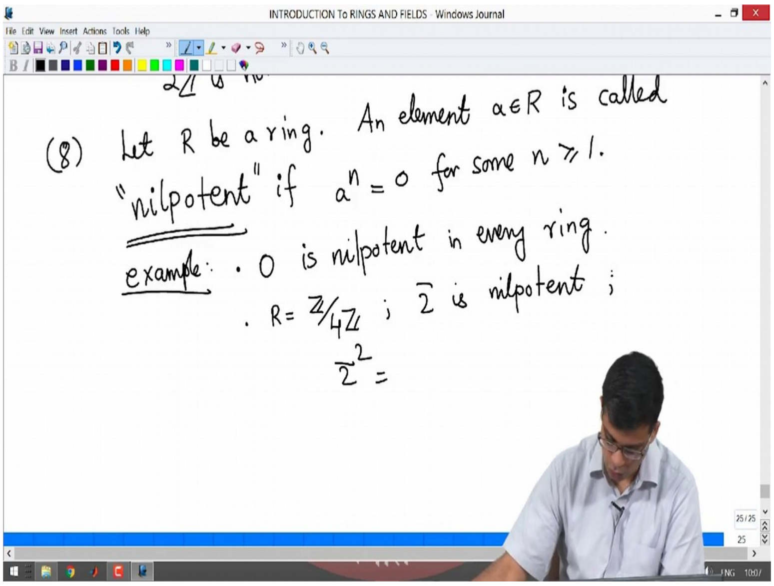

So, the number 8. I want to talk about the following. So, let R be a ring. An element of R, an element a of R is called I will say "nilpotent". So, this is a new word for us "nilpotent", if a power n is 0 for some n ok. So, this is what a nilpotent element is. So, this is an important, this is an important property in ring theory. So, a nilpotent element is one such that a power of it is 0. For example, 0 element is nilpotent always because 0 power 1 is 0, 0 is nilpotent in every ring, right.

There are other nilpotent elements in specific examples. Let us take R to be, let us say Z mod 4Z. So, here 2 bar is nilpotent. Why is this? Why is 2 bar nilpotent? 2 bar is nilpotent because, 2 bar squared is 2 bar is of course not 0; so, this examples shows that there are nilpotent elements that are not 0.

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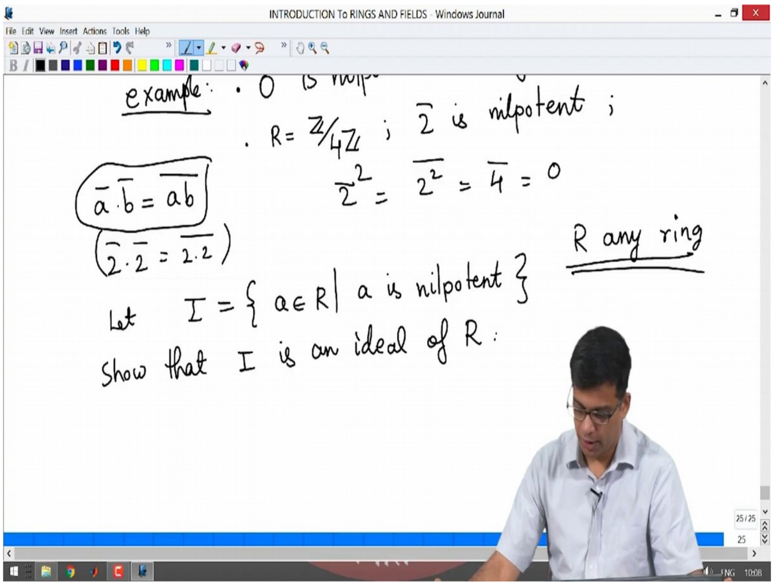

So, 2 bar squared is 2 squared bar, right, this is how we multiply in Z mod 4Z. So, a bar times b bar is a b bar. So, 2 bar times 2 bar is 2 times 2 bar ok, but that is 4 bar. What is 4 bar? 4 bar is 0, right, because we are going modulo 4. So, 2 bar is a nilpotent element. So, the problem that I want to do now is, consider the set of all nilpotent elements. So, let I be a in R such that a is nilpotent. So, R is any ring here. So, show that I is an ideal. So, show that I is an ideal of R. So, that is the problem.

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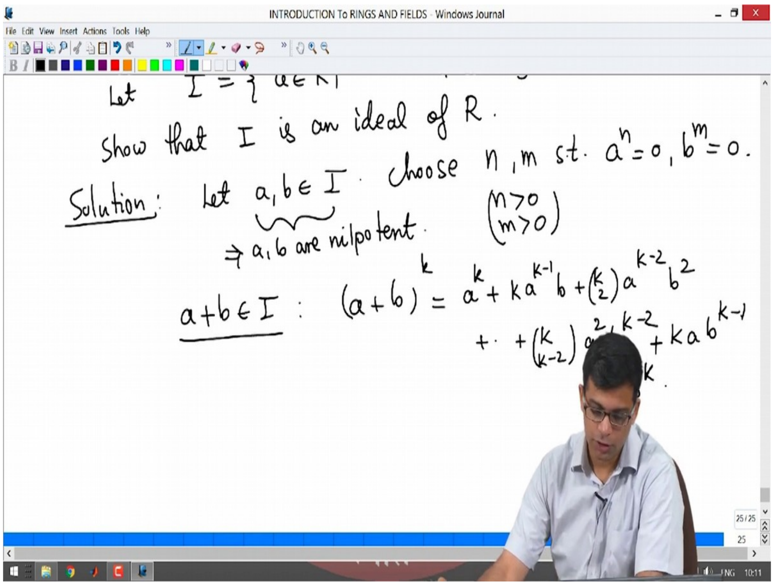

So, solution; so, I will not do the details fully. This by now you are comfortable with working with elements and ideals and so on. So, we can complete this. Why is this an ideal? So, again to check something is an ideal, we have to check that it is closed under addition, closed under arbitrary ring multiplication. So, let us say a and b are in I that means, a and b are nilpotent.

So, choose n, m, such that a power n is 0 and b power m is 0, such things exist because a and b are nilpotent. So, by definition some power of a; so n is of course a positive integer, m is also a positive integer. The definition is a power n is 0 for some positive integer n. So, a power n is 0 for some positive integer n, b power m is 0 for some positive integer m ok.

So now, if you want to show that a plus b is in I, what I will do is, I want to rise it to some large power. So, I want to rise it to some power to make it 0 ok. So now, suppose I choose a power. What is a power? So, if I put a power k here, what is this? By binomial theorem which applies in a ring because you can use distributive property of addition and multiplication by in order to split this up into different factors and multiply and then at the end put them together.

This will be something like a power k plus k times a power k minus 1 b and then k choose 2 a power k minus 2 b squared and so on. You will have k choose k minus 2, a squared b power k minus 2 plus k a b power k minus 1 finally, b power k, right. This is what it would be.

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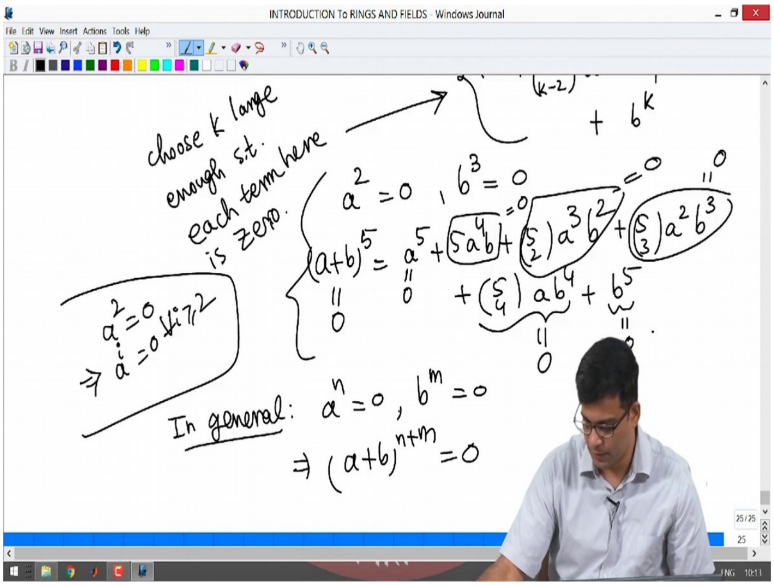

So, can we choose k large enough, large enough such that each term here is 0. So, because we know that a power of a is 0. For example, let us say a squared is 0 and b cubed is 0. Now, if you do a power b a plus b times a plus b power, ok. So, let us say maybe 5 will work. So, what is this? This is a power 5 plus 5 a power 4 b plus 5 choose 2. So, that is a 10 actually, but a cubed b squared plus 5 choose 3 a squared b cubed plus 5 choose 4 a b 4 plus b 5 ok.

So, this is 0, so if a squared is 0 it is very easy to check that a power n is 0 or a power i is 0 for all i greater than equal to 2 right, because a squared is 0 means a time a squared times a is 0 that means a cubed is 0, a to the forth is 0 and so on. So, a power 5 is 0, this is also 0 because if no matter what b is a power 4 is 0. This is 0 because a cubed is 0 again. This is 0 of course, because a squared is 0. Now, we use the fact that b cubed is 0, so this is 0 and this is also 0. So, a power a plus b power 5 is 0.

So, this idea will tell me that; so, this is an example. So, in general a power n is 0, b power m is 0 implies a plus b power n plus m is 0 ok. Maybe even you can actually choose a smaller exponent perhaps, but I do not really care about that.

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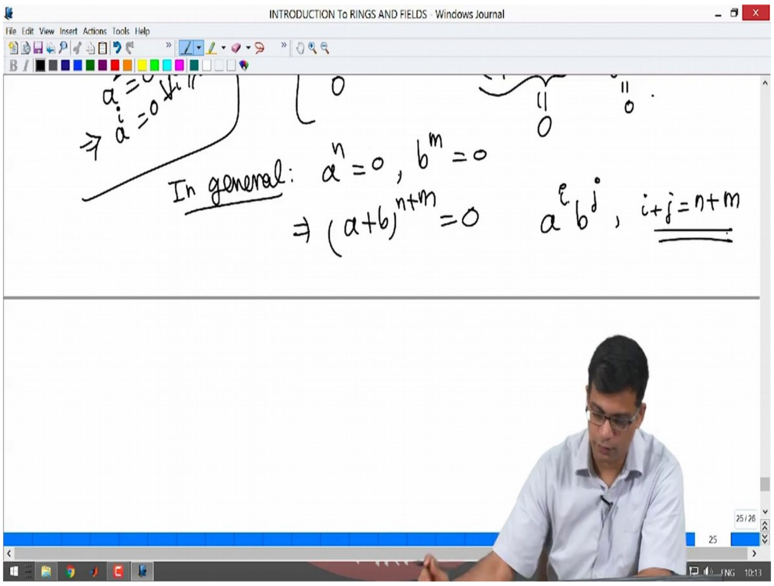

This is because when you expand this out each term each binomial term will be a power I, b power J. Either I plus I will be at least n or J will be at least m because if they are both strictly less than n and strictly less than m than the sum cannot be, sum has to be n plus m, right. So, when binomial coefficient here each sum is, the exponents will add up to k, k minus 1 plus 1, k plus 0, k minus 2 plus 2. So, a plus b will add up, I plus J will add up to n plus m. So, either I will be at least n or J will be at least m.

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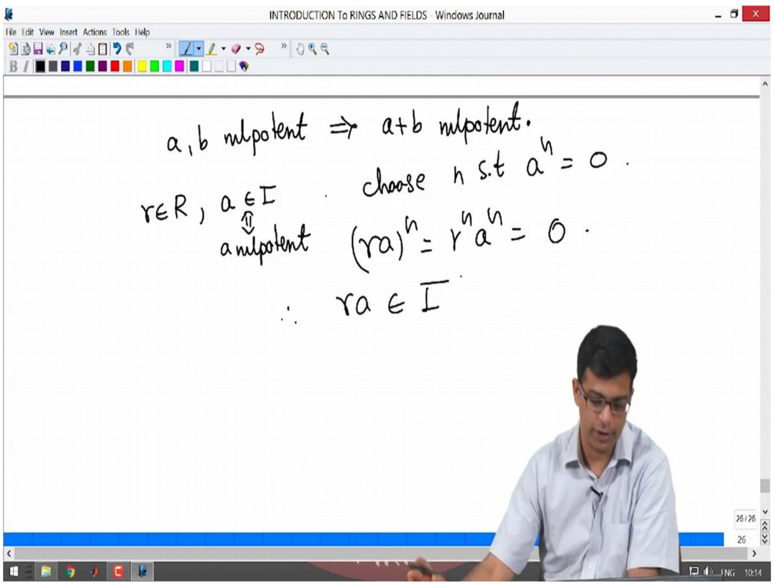

So, if a and b are nilpotent, what I have just shown is, if a and b are nilpotent that means, a plus b is nilpotent ok. So, this is good. So, and also, we need to check now that if r is R and a is in I that means, a is nilpotent. Remember, I is the set of nilpotent elements, I is the set of all nilpotent elements. Then, so choose n such that a power n is 0, by definition such an n exists. Now, if you take r a power n, this is r n a n, this is 0. So, r a is also nilpotent, so it is in I.

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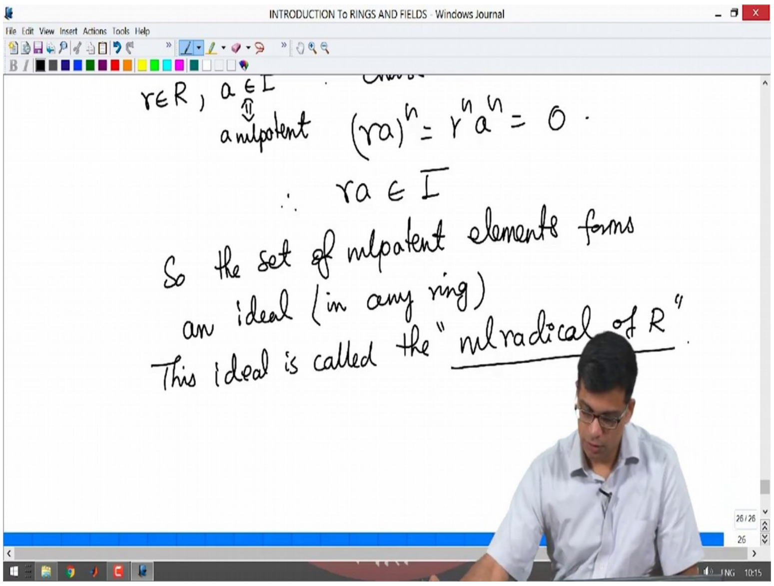

So that means, so the set of nilpotent elements forms a ring, sorry, forms an ideal in any ring here. So, R is an arbitrary ring. So, this is called, this ideal is called the "nilradical of R" ok. So, just to complete this example, the nilradical is an important ideal in a ring. What is the nilradical of Z?

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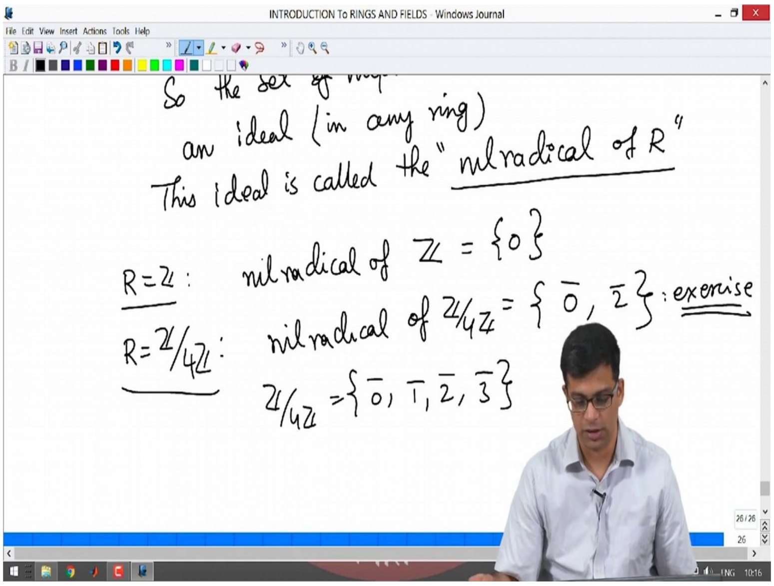

So, if you think about this, here if you take an integer and some power is 0, that means that integer must be 0 itself ok. So, nilradical must be the 0 just the 0 ideal. And as the earlier example showed if R is Z mod 4Z, what is nilradical? We already saw that, 0 remember is always in any ring, 0 is an nilradical; is in the nilradical because its nilpotent, but 2 bar is also an a nilpotent element. So, Z mod 4Z of course, is 0 bar, 1 bar, 2 bar, 3 bar, 1 bar and 3 bar are not nilpotent you can check that and clearly 0 bar and 2 bar are nilpotent. So, this equality I will that as an exercise.

So, hopefully these examples and exercises gave you some comfort with working with elements of a ring, ideals of rings and so on. So, in the next video we will continue our study of ring theory.

Thank you.