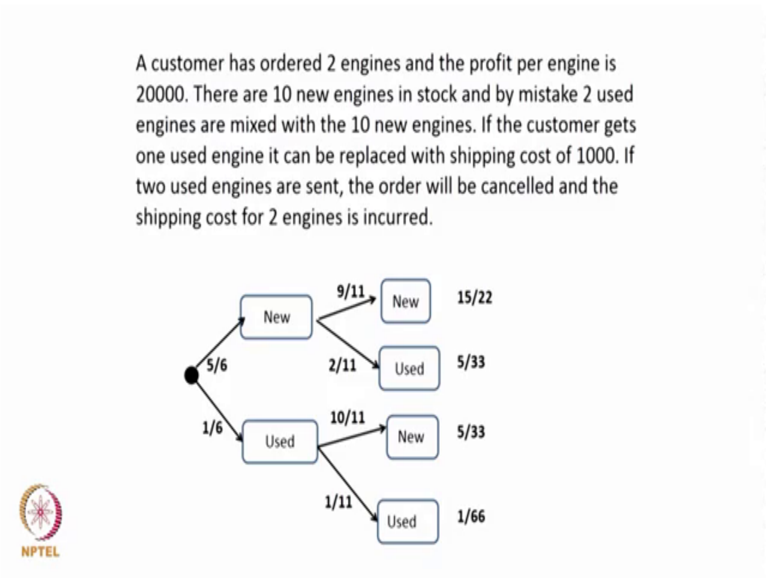
Introduction to Probability and Statistics
Prof. G. Srinivasan
Department of Management Studies
Indian Institute of Technology, Madras

Lecture – 17
Random Variables – concepts and exercise

In this lecture we continue the discussion on Random Variables.

(Refer Slide Time: 00:24)



In the previous lecture, we were looking at this problem where we said that; the customer has ordered 2 engines and then we looked at a situation where, the 10 new engines and 2 old engines which have been added by mistake. And then we tried to find out what is the expected value and we also found out what is the standard deviation of X.

(Refer Slide Time: 00:41)

Outcome	Gain (x)	Deviation (x – μ) (μ = 33007)	Squared deviation $(x-\mu)^2$	Probability (X = x)
Both new	40000	6993	48902049	0.682
One used	19000	-14007	196196049	0.303
Both used	-2000	-35007	1225490049	0.015

E(X) = 40000 x 0.682 + 19000 x 0.303 – 2000 x 0.015 = 33007

$$Var(X) = 111180951;$$

$$SD(X) = 10544.24$$

Though the expected value is high, the high SD indicates that there can be loss

(Refer Slide Time: 00:44)

Properties of expected values

Adding or subtracting a constant

E(X ± c) = E (X) ± E (c) = E (X) ± c

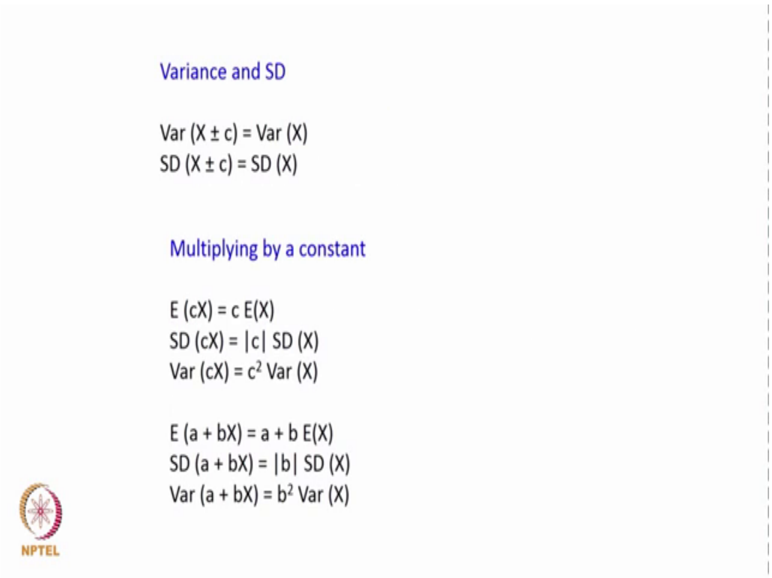
Expected value after paying the initial shipping fee is 33007 – 1000 = 32007.

(Capital letters denote random variables. Lower case letters indicate constants or values that it can take)

Now, let us continue the discussion by looking at properties of expected values, because we always have this question you know; what happens when we add or subtract a constant, what happens when we multiply and so on.

So, expected value of X plus or minus c, where c is a constant is E of X expected value of X plus minus expected value of c. Now c being a constant it is expected value will be c itself and therefore, E of X plus or minus c is equal to E of X, E of X plus or minus c. One has to read this carefully this is E of expected value of X plus or minus constant is equal to expected value of X plus or minus constant. For example, in the previous problem, what is the expected value after paying the initial shipping fee; that would be if the shipping fee is 1000 then 33007 minus 1000 is 32007. Also we have to remember that capital letters X represents a random variable, and lowercase letters generally indicate either constants or the values that the random variable can take. Therefore, you will always find you know P of X equal to X capital X equal to small x.

(Refer Slide Time: 02:10)



So, capital X is the random variable which takes a value small x. Now what happens to the variance and standard deviation. So, variance of X plus minus c which means when we add or subtract a constant, the variance does not change. So, it remains as variance of X. Similarly, standard deviation X plus minus c also will not change because the variance had not changed, and standard deviation of X plus or minus c is equal to standard deviation of X. If we multiply by a constant then expected value of c X, where c is the constant is c times expected value of x.

So, the expected value of X gets multiplied by c whereas, the left hand side expected value of multiplying every possible value that capital X can take by c. Now standard deviation of c X is equal to positive value or absolute value of c because a time c can be negative. So, absolute value of c into standard deviation of X, and variance of c X is equal to c square into variance of X. And since there is a square root involved it will become standard deviation has to be positive therefore, we take absolute value of c.

Now, what happens when we have a combination of an addition and a multiplication. So, E of a plus b X is equal to a plus b times E of X. Now that is a direct application of what we saw here, and what we saw here. So, expected value of a plus b X is equal to a plus b times expected value of X. Standard deviation of a plus b X is equal to absolute value of b, remember the a will go; a being a constant will not have an effect in the variance and in standard deviation so a goes. Only b is important and b can be negative so we also want to say here that we take the positive value of the b.

So, absolute value of b into standard deviation of X and variance of a plus b X will be b square into variance of X.

(Refer Slide Time: 04:18)

Exercise

A and B play a game. A and B toss coins. If both are heads A gets Rs 200; if both are tails B gets Rs 100; and if it is one head and one tail no money is given. A has a biased coin with 60% probability of heads while B has a fair coin. Find E (A) and E (B) and the variances

Outcome	Gain (A) (x)	Deviation $(x - \mu)$ $(\mu = 40)$	Squared deviation $(x - \mu)^2$	Probability $(X = x)$
Both H	200	160	25600	0.3
H/T T/H	0	-40	1600	0.5
Both T	-100	-140	19600	0.2

E (A) = 40; Var (A) = 12400; SD (A) = 111.36

We now look at this exercise where A and B play a game, A and B toss coins. If both are heads that is if both A and B toss heads A wins rupees 200, if both are tails B wins rupees 100. And if one is a head and one is a tail no money is transacted. We also assume that A has a biased coin with a 60 percent probability of heads, while B has a fair coin with a 50 percent probability of heads. Find the expected value of A expected value of B and the variances.

Now, this random variable there are 3 outcomes both toss heads, one head one tail or one tail one head; and both toss tails. So, we look at this problem from as perspective. So, gain for person A when both toss heads is 200 with a probability of 0.3, because A tossing a head is 0.6 the problem assumes that a has a biased coin. B has a fair coin with

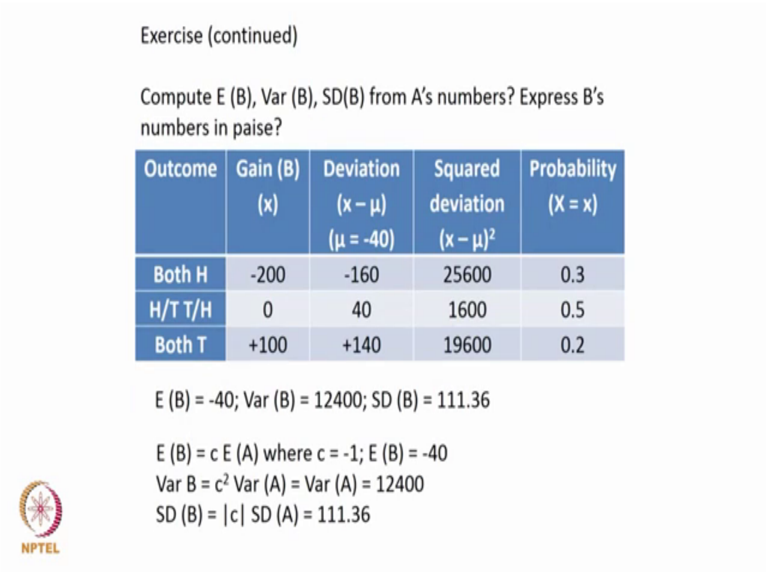
0.5 therefore, both tossing heads is 0.6 into 0.5 which is 0.3. One head one tail is a tossing A head 0.6 B tossing at a tail 0.5 which gives us 0.3. A tossing a tail 0.4 B tossing a head 0.5 multiplication is 0.2.

So, 0.3 plus 0.2 is 0.5 which is shown here, H T and T H is 0.5, H T is 0.3, T H 0.2 and the total is 0.5. Both tossing tails A tosses tail with 0.4, B tosses tail with 0.5 and therefore, both tossing tail is 0.4 into 0.5 which is 0.2 which is also given here. Now the expected value is 200 into 0.3 plus 0 into 0.5 minus 100 into 0.2.

So, 200 into 0.3 is 60, 0 into 0.5 is 0, minus 100 into 0.2 is minus 20 therefore, the expected value is 40. Now this expected value is shown as mu equal to 40 here. So, we find the deviations X minus mu 160 minus 40 and minus 140. Square values are shown 25600, 1600, 19600.

So, the variance is 25600 into 0.3, plus 1600 into 0.5, plus 19600 into 0.2 which gives us 12400 and standard deviation of A is square root of 12400 positive square root which is 111.36.

(Refer Slide Time: 08:06)



Exercise (continued)

Compute E (B), Var (B), SD(B) from A's numbers? Express B's numbers in paise?

Outcome	Gain (B) (x)	Deviation $(x-\mu)$ $(\mu = -40)$	Squared deviation $(x-\mu)^2$	Probability $(X = x)$
Both H	-200	-160	25600	0.3
H/T T/H	0	40	1600	0.5
Both T	+100	+140	19600	0.2

E (B) = -40; Var (B) = 12400; SD (B) = 111.36

E (B) = c E (A) where c = -1; E (B) = -40
Var B = c^2 Var (A) = Var (A) = 12400
SD (B) = |c| SD (A) = 111.36

Now let us look at the problem from B's point of view, and try to understand the relationship between the expected value of B and the variance of B versus expected value for A and the variance for A. Now when we look at this from B's point of view again the random variable takes 3 values with a difference; that when both tasks heads B loses 200 therefore, B the value that it takes for both heads is minus 200 with the same probability of 0.3.

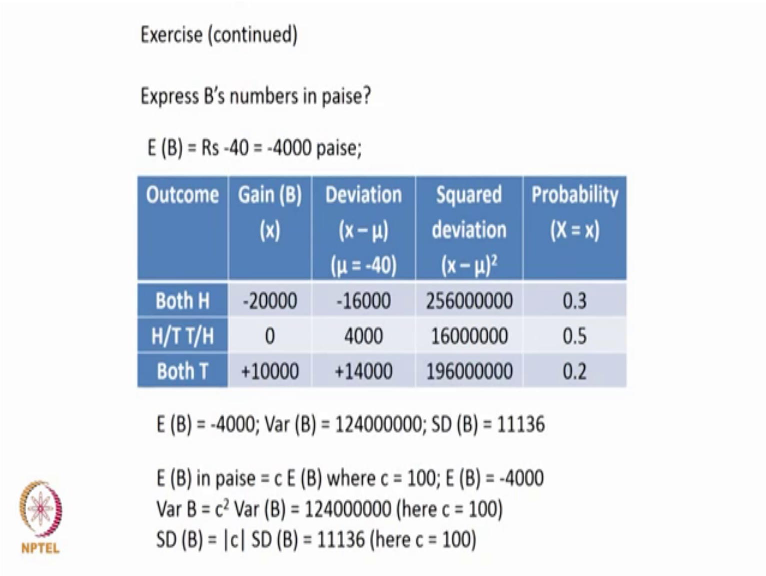
So, when A gains 200 B loses 200 therefore, we get minus 200 with 0.3 0 with 0.5 and plus 100 with 0.2. Remember that for both tails A lost 100 which is B's gain and therefore, B gets plus 100. Now the expected value of this random variable is minus 200 into 0.3 plus 0 into 0.5 plus 100 into 0.2, which is minus 40. Variance of B is we find out the deviations so mu is shown here as minus 40. So, the X minus mu for the 3 outcomes are minus 200 minus minus 40 which is minus 160. Once again minus 200 minus minus 40 is minus 160, 0 minus 40, 0 minus minus 40 is 40, 100 minus minus 40 is 140. Please note that mu is minus 40 and therefore, the calculations are shown like this.

If you see carefully in the previous case the value is where 160 minus 40 and minus 140 here it is minus 160, 40 and 140. So, X minus mu square will be the same because here we have negative there you had a positive so 25600, 1600, 19600. So, the variance is 25600 into 0.3 plus 1600 into 0.5 plus 19600 into 0.2, which is 12400 and the standard deviation is 111.36. So, up to this point we realize that expected value of B is the negative of the expected value of A. Variance of B is the same, standard deviation of B is also the same, from the earlier relationships we saw that E of B is equal to c times E of A.

Now, c is minus 1 because what is A's gain is B's loss and what is A's loss is B's gain. Therefore, if E of A is 40, E of B is c times expected value of a minus 1 into 40, which is minus 40, which is what we computed here. Variance of B is c square into variance of a where c is minus 1, c square is plus 1. So, variance of B is equal to variance of A and we found out that both are 12400, for A it is 12400 for B also it is 12400. Standard deviation of B is absolute value of c into standard deviation of A, since c is minus 1 absolute value is plus 1 so one into standard deviation of A same 111.36 and both show the same value.

Therefore, we have just shown and demonstrated this equation that E of B is equal to c times E of A and variance is the same in this case; where c is minus 1 c square is plus 1. So, variance of B is c square into variance of A and standard deviation of B is absolute value of c into standard deviation of A.

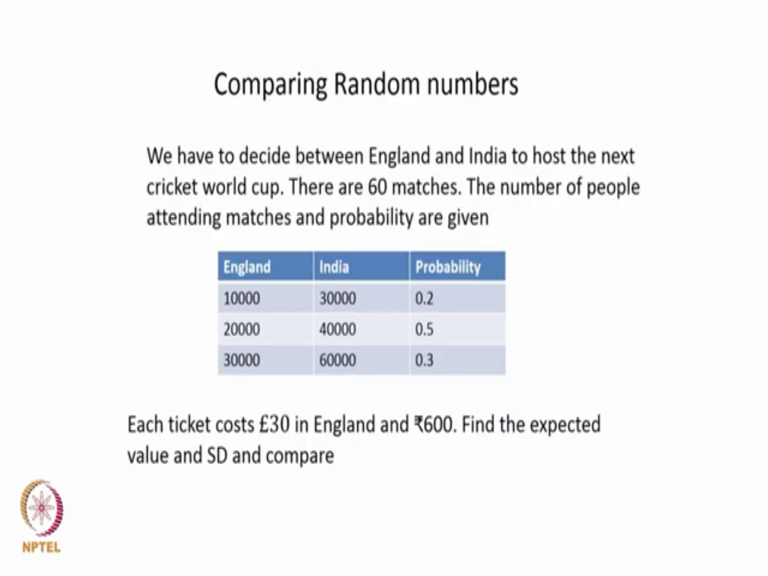
(Refer Slide Time: 12:32)



Now if we express these numbers in paise, what happens to the mean and the variance? Now, expected value of B is rupees minus 40 which is minus 4000 paise. And therefore, the 3 gains are minus 200 becomes minus 20000, 0 stays as 0 and plus 100 becomes 10000. So, X minus mu gets multiplied by 100 earlier it was minus 160, 40 and plus 140 now it is minus 16000 plus 4000 and plus 14000. The squares get multiplied by 100 into 100 which is 10000. So, 256 1 2 3 4 5 followed by 6 0's and 16 followed by 6 0's and 96 followed by 6 0s. And therefore, the variance is 256000000 into 0.3 and so on. And when we do this we get 124000000, which means the variance gets multiplied by 10000 and the standard deviation gets multiplied by 100, the expected value also gets multiplied by 100.

So now we try to establish that relationship. So, expected value of B in paise is c times the normal E of B, now in this case c is 100. So, earlier expected value of B was minus 40 now it becomes minus 4000, which we have calculated here which we can also see as minus 20000 into 0.3 plus, 0 into 0.5, plus 10000 into 0.2. So, variance of B in paise is equal to c square into variance of B in rupees. So, 12400 becomes 12400 into 10000. So, since c is 100 c square is 10000 it gets multiplied by 10000. Standard deviation of B is equal to absolute value of c into standard deviation of B in rupees. So, standard deviation of B in paise is 100 times standard deviation of B in rupees which is 111.36 into 100 which is 11136.

(Refer Slide Time: 15:06)



Now let us look at another example to compare random numbers and understand this. So, let us take an imaginary situation where, say we have to decide between England and India to host the next cricket world cup. Now let us say there are 60 matches the number of people attending matches and the probability are given. So, please note that this is an imaginary problem that we are trying to solve, and we are only trying to explain the methodology using this interesting imaginary situation.

So, we would say that in England the expected people could be 10000 or 20000 or 30000 attending a match whereas, in India it could be 30000, 40000, 60000 attending a match, and with probability is given as 0.2 0.5 0.3. Let us assume the ticket cost about 600 in India and 30 pounds in England find the expected value standard deviation and compare.

(Refer Slide Time: 16:06)

England

Outcome	Deviation	Probability (X = x)
10000	-11000	0.2
20000	-1000	0.5
30000	9000	0.3

E (Eng) = 21000; Var (Eng) = 49000000; SD (Eng) = 7000

India

Outcome	Deviation	Probability (X = x)
30000	-14000	0.2
40000	-6000	0.5
60000	16000	0.3

E (Ind) = 44000; Var (Ind) = 482000000; SD (Ind) = 21954

So, let us say we do it in England so outcome 10000 people attending with probability 0.2 and so on. So, expected value for the people attending is 10000 into 0.2 plus 20000 into 0.5 plus 30000 into 0.3, and the expected value is 21000; now a deviations are also given so minus 11000 minus 1000 and 9000.

So, we find the variance and we find the standard deviation. Now, for the same thing done in India 30000 with a probability 0.2 and so on. So, expected value is 44000 and the variance and standard deviation are bigger figures.

(Refer Slide Time: 16:50)

England

E (Eng) = 21000; Var (Eng) = 49000000; SD (Eng) = 7000
E = 37800000 £ SD = 1626653 £

India

E (Ind) = 44000; Var (Ind) = 482000000; SD (Ind) = 21954
E = 1584000000 ₹ SD = 102032972

Converting 1$ = 0.72 pound = 68.67 Rs

E (England): 52500000$ SD = 2259240 Coefficient of variation = 0.043
E (India): 23066841 SD = 1485845 CV = 0.0644

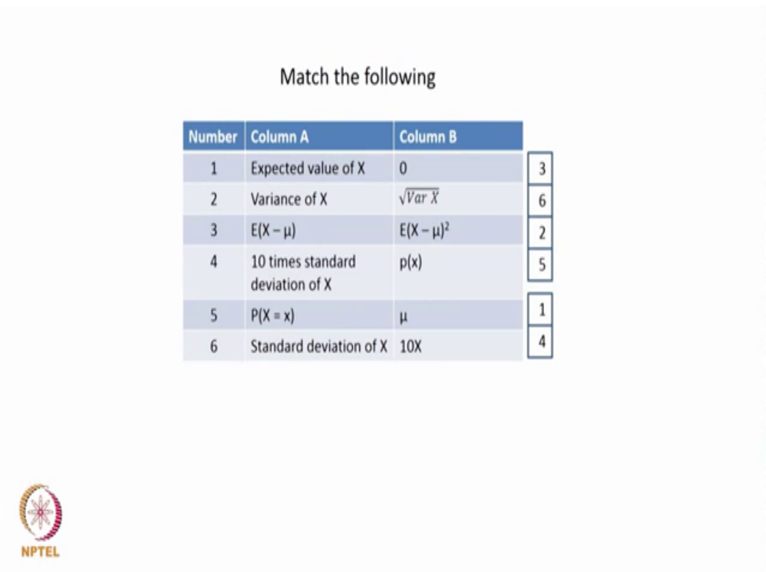
Choose lower CV

So now, if we look at place one which is England and then the expected value is 21000 variance are given. And let us say we also try to convert the whole thing is in pounds. So, standard deviation is 1626653.

Now, in India so we multiply by the money and therefore, when we multiply by the money we get this in pounds which is given here, and this is given here. So, 21000 people 378000 pounds and so on, with the variance and standard deviation are given here. Now for India we converted into rupees we use that 6000. So, the variance and standard deviation figures are given here in rupees. And then we convert one dollar equal to 0.72 pounds and 1 dollar is 68.67 rupees to make a comparison in a third currency, just to understand the idea of multiplying. And then we realize that for England expected value is 5250000 dollar, and standard deviation with the coefficient of variation of 0.043. In India the coefficient of variance is 0.044 and therefore, we can try to choose the place that has a lower coefficient of variation.

So, that way this example helps us to understand the multiplication, the comparisons, the change in currency and so on. So, while the number of people attending was the random variable that was later converted to the money generated. So, all these can be done and decisions can be made against a common denomination. And that is what this example essentially tells us. Now let us continue this with a discussion on the random variables.

(Refer Slide Time: 18:42)

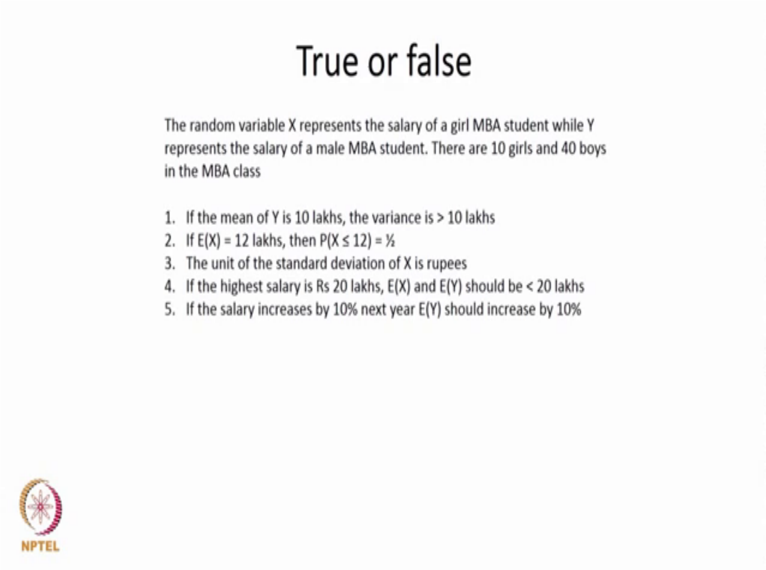


Match the following

Number	Column A	Column B	
1	Expected value of X	0	3
2	Variance of X	$\sqrt{Var\ X}$	6
3	$E(X-\mu)$	$E(X-\mu)^2$	2
4	10 times standard deviation of X	p(x)	5
5	$P(X = x)$	μ	1
6	Standard deviation of X	10X	4

So, we begin with a simple match the following example. So, there are 6 items expected value of X takes mu which is here. So, which is given by this, expected value of X is mu. Variance of X is the expected value of X minus mu the whole square. Expected value of X minus mu will be 0, 10 times standard deviation of X is 10 X if X is the standard deviation. Probability of capital X equal to small x is P of X, and standard deviation of X is the root of the variance of X. So, that is how we match and try to understand the relationship between the items and column A and the items in column B.

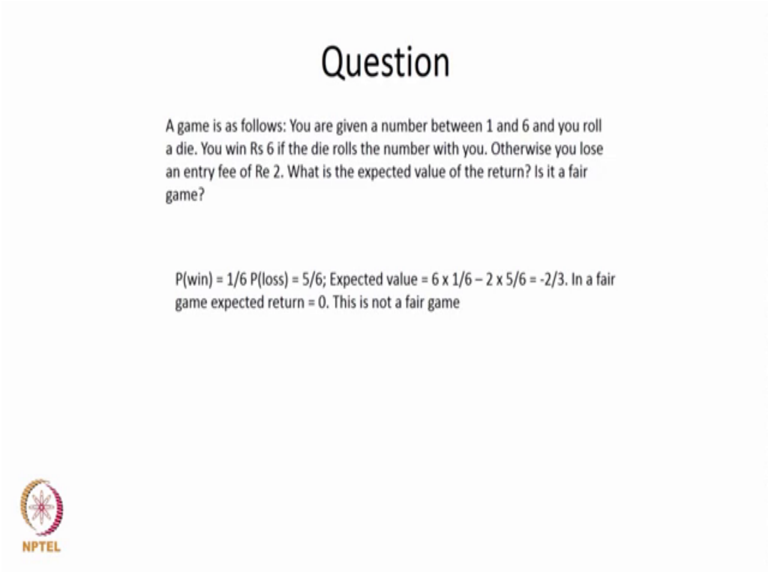
(Refer Slide Time: 19:27)



Now, let us look at this the random variable X represents the salary of a girl MBA student, while the random variable Y represents the salary of a male MBA student. There are 10 girls and 40 boys in the MBA class. If the mean of Y is 10 lakhs, the variance is greater than 10 lakhs. Need not be true at all we could have a situation where the y is actually the salary of the male. So, we could have a situation where the variance is greater than 10, and we could have a situation where the variance is less that. If expected value of X is 12 lakhs then p of X less than 12 is half need not be probability of X taking a value of 12 is half need not be again.

The unit of standard deviation of X is rupees could be true, because the salary right now has been given in terms of lakhs of rupees. So, we can convert it to rupees and then measure the standard deviation of X in rupees. If the highest salary is 20 lakhs then E of X and E of Y should be less than 20 lakhs. We could say true the only case that will happen is all of them have the same 20 lakhs, in which case there will be equal to 20 lakhs. So, we can generalize it and say E of X and E of Y should be less than or equal to 20 lakhs is true. If the salary increases by 10 percent for everybody, E of Y should also increase by 10 percent yes E of Y would also increase by 10 percent.

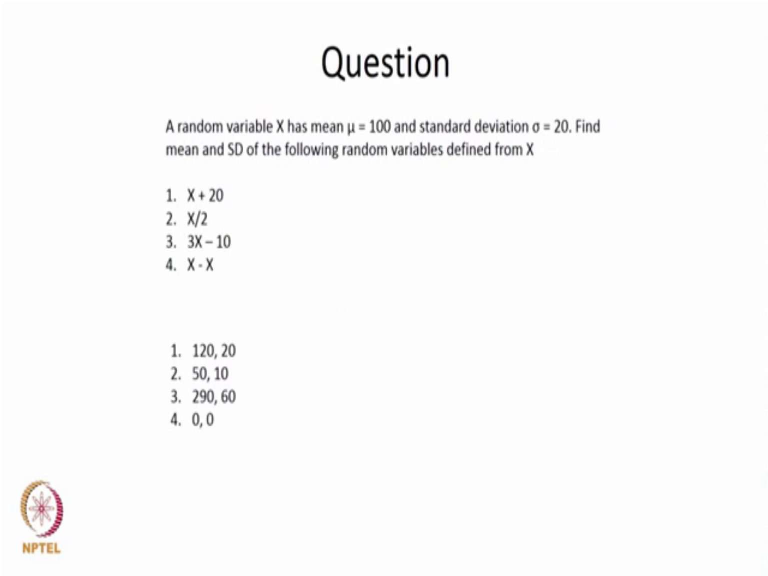
(Refer Slide Time: 21:01)



Now, we look at another question a game is as follows. You are given a number between 1 and 6 and you roll a die. You win rupees 6 if the die rolls the number that is with you; otherwise you lose the entry fee of rupees 2, what is the expected value of the return is it a fair game. So, probability of win is 1 by 6, because you have a number that die can roll any 1 of the 6 numbers. So, probability that the die rolls the number that is with you is 1 by 6. So, you win with a probability of 1 by 6 and you lose with a probability of 5 by 6. So, expected value is 6 into 1 by 6, I get 6 rupees we I win. So, 6 into 1 by 6 and I lose 2 rupees so minus 2 into 5 by 6 which is minus 2 by 3 of course, one can argue that even I win I actually have an entry fee of 2 therefore, my gain is only 4, but right now we use 6 into 1 by 6. Let us assume 6 is the gain which also means that you get 6 rupees more than the entry fee.

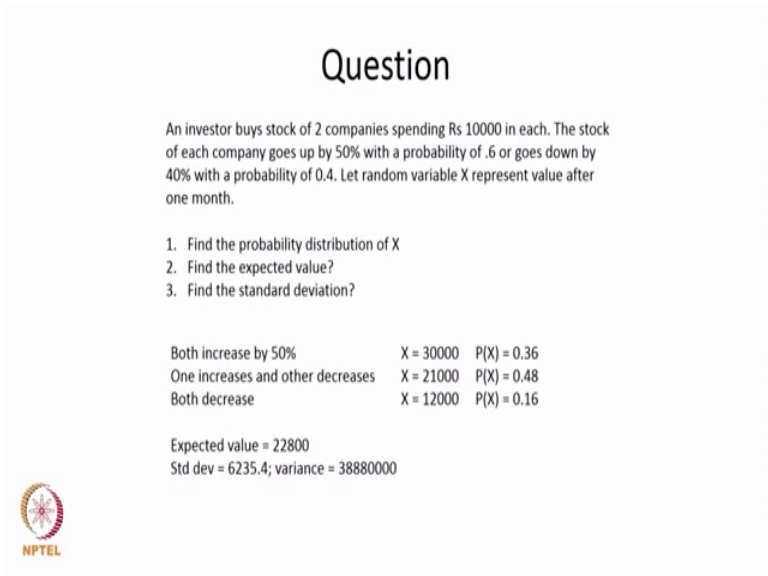
So, 6 into 1 by 6 minus 2 into 5 by 6 which is minus 2 by 3. So, in a fair game the expected return is 0 and therefore, this game is not a fair game.

(Refer Slide Time: 22:27)



Now a random variable has mean mu equal to 100 and standard deviation sigma equal to 20. Find the mean and standard deviation of the following random variables defined from X, X plus 20. So, X plus 20 the mean changes from 100 to 120 the standard deviation remains at 20 and does not change. X by 2, the expected value also becomes half which is 50 standard deviation becomes half which is 10. 3 X minus 10 expected value will be 3 times 100, 300 minus 10 to 90. The minus 10 does not have an effect on the standard deviation therefore, it will be only multiplied by 3 which is 60. X minus x so mean is 0 and standard deviation also will be 0.

(Refer Slide Time: 23:19)



Investor buy stocks of 2 companies spending 10000 in each, the stock of each company goes up by 50 percent with a probability of 0.6 or goes down by 40 percent with the probability of 0.4. So, let it the random variable X represent the value after one month. So, both decrease, both increase by 50 percent X is 30000 probability 0.36. One increases and the other decreases X is 21000 probability of X is 0.4 8, and both decrease probability is 0.16. So, expected value is 22800, standard deviation is 6235.4. So, with this we come to the end of our discussion on random variables. And we will continue our discussion on models on probability in the next lecture.