Numerical Analysis
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Lecture No 5
Part 2
Lagrange Interpolation Polynomial Error in Interpolation 1

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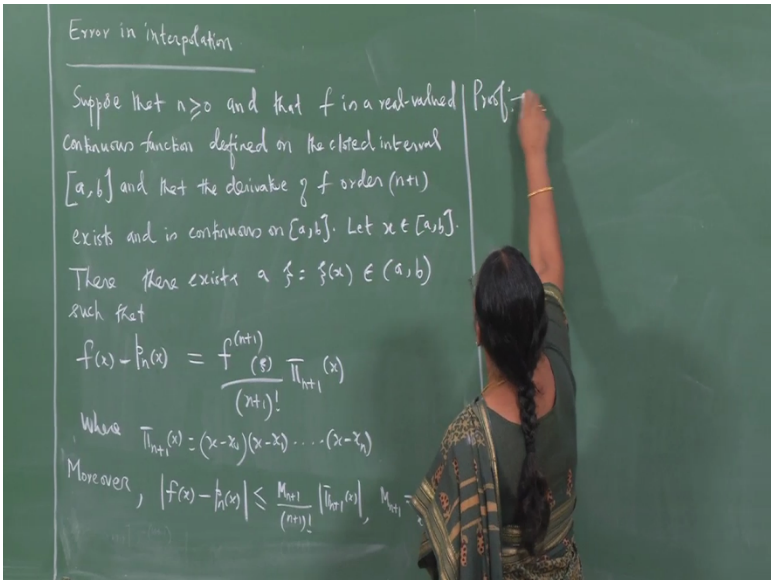

We consider now error in interpolation, suppose that n is greater than or = 0 and that f is a real value continuous function defined on the closed interval a, b and that the derivatives of f of order n + 1 exist and is continuous on the closed interval a, b. Let x belong to the close interval a, b, then there exists a Xi which depends on x belonging to the open interval a, b such that f(x) + P n of x the interpolating polynomial of degree at most n that interpolates this function f(x) this, the n + 1 derivative of Xi by n + 1 factorial into Pi n + 1 of x.

Where Pi n + 1 of x is x + x 0 x + x 1 etc up to x + x n this is a polynomial of degree n + 1 there are n + 1 such factors of the form x + x 0 x + x 1, etc, x + x n. So Pi n + 1 of x is a polynomial of degree n + 1. Moreover, we will show that modulus of f(x) + P n of x is less than or = M n + 1 by n + factorial into modulus of Pi n + 1 of x, where M n + 1 = maximum of modulus of n + 1 derivative of f for x in the interval a, b, this gives you the error bound on the interpolation error. We shall now provide details of the proof of this theorem.

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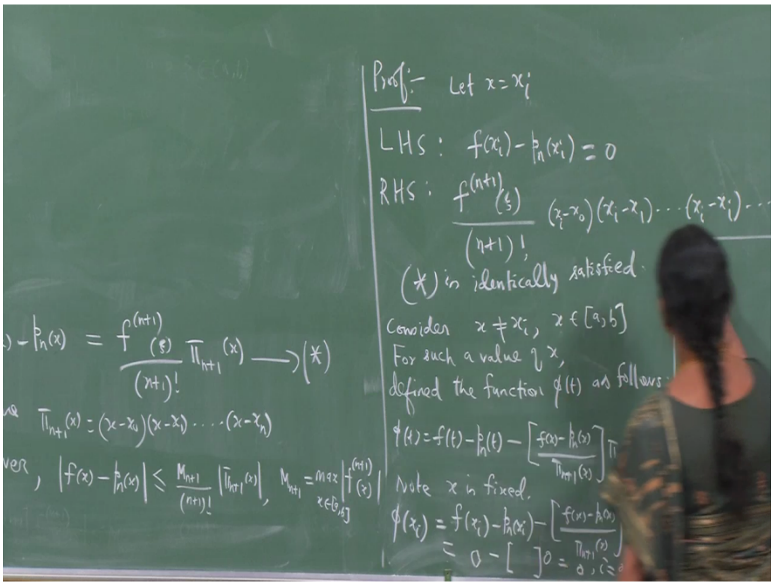

Let us first take x to be say x i, what are x i? x is are the interpolation point, so what happens to the left hand side of star? Suppose I call this a star, then the left-hand side is such that I have f(x i) + P n of x i and what is it, that is 0 because x i have the interpolation points at which P n of x interpolates the function f, so the left-hand side is 0. What happens to the right hand side, it is going to be n + 1 derivative of Xi by n + 1 factorial into Pi n + 1 of x. What is Pi n + 1 of x? It is x + x 0 and I take x to be x i, so x i + x 0 x i + x 1 etc, and there will be a factor x i + x i into x i + x n. So because of this factor this is 0 so the right hand side is also 0, so star is identically satisfied at points x which are x i namely the interpolation points, so star is identically satisfied when x is x i.

So we consider the case when x is different from xi and x belongs to the interval a, b. So for such a value of x, we define the function Phi of t as follows; namely Phi of t is defined as f(t) + P n of t + f(x) + P n of x by Pi n + 1 of x into Pi n + of t. 0e that here x is fixed and x belongs to the interval a, b. Now I shall look into properties of Phi F(t), Phi at x i where x i are the interpolation points is such that it is f of x i + P n of x i + f(x) + P n of x divided by Pi n + 1 of x into Pi n + 1 at x i. But what do I know about f of x i + P n of x i that is a 0 + the terms within this bracket is a constant because x is fixed. What about Pi n + 1 of x i? Pi n + 1 of xi contains the factor x i + x i in it and hence that is 0 and shows that Phi of xi is 0 for i = 0, 1, 2, 3, up to n, so there are n + 1 points xi at which Phi vanishes.

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In addition let us see what is Phi at x where x belongs to a, b? So Phi at x will be f of x + P n of x + f of x + P n of x by Phi n + 1 of x into Phi n + 1 of x, so we get f of x + P n of x + f of x + P n of x and so that is again 0. So we observe that Phi vanishes at point x, x 0, x i, x 2, etc, up to x n, so there are n + 2 points at which Phi vanishes. So applying Rolle's Theorem we see that Phi dash of t vanishes at n + 1 distinct points and each of these points' lies between the successive 0s of Phi. I can continue to apply Rolle's Theorem again and that gives me that Phi double dash of t vanishes at n distinct points, each of which lies between the successive 0s of Phi dash.

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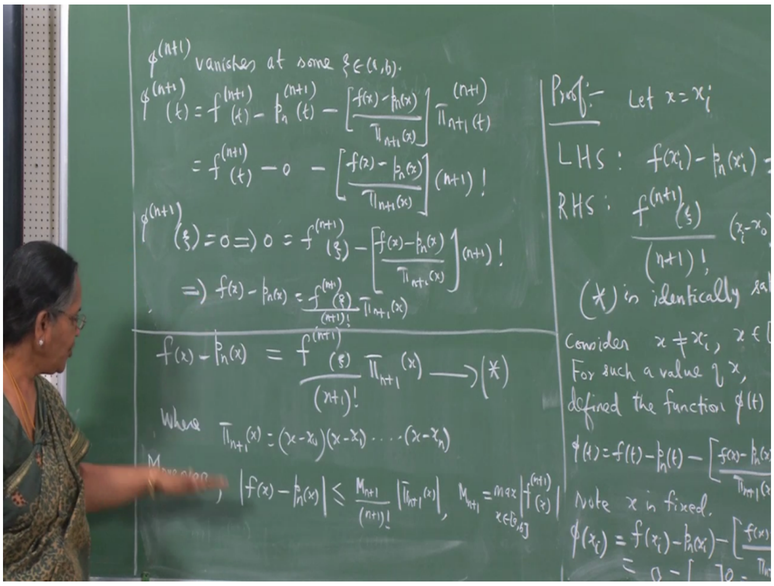

The assumptions about the function f in the theorem are sufficient, so that Rolle's Theorem can be applied in succession n + 1 times and that shows that the n + 1 derivative vanishes at some Xi in the interval a, b, so we compute the n + 1 derivative of Phi. So the n + 1 derivative of Phi will be the n + 1 derivative of f + the n + derivative of P n + f(x) + P n of x by Pi n + 1 of x into the n + 1 derivative of Pi n + 1 of t.

So this will be n + 1 derivative of F Phi t, what do you know about P n? It is a polynomial of degree at most n, so n + 1 derivative is 0 so this term will give you f(x) + P n of x by Pi n + 1 of x into, let us find out what is the n + 1 derivative of Pi n + 1. We know that Pi n + 1 of x is a polynomial of degree n + 1 and the leading term is x to the power of n + 1. So when you differentiate this n + 1 times that will give you n + 1 factorial.

But what do we know application of Rolle's Theorem in succession n + 1 times gives that n + 1 derivative of Phi vanishes at some Xi. So the n + 1 derivative of Xi is 0 and that gives you that n + 1 derivative of f at Xi + f of x + P n of x by Pi n + 1 of x into n + 1 factorial is 0. So if you the rewrite this, we get f(x) + P n of x to be n + 1 derivative of f at Xi by n + 1 factorial into Pi n + 1 of x and this is what we are asked to show namely result star. So we have shown that the error in interpolation at any x in the interval a, b which is different from x i is given by the right hand side, so we now have to prove this result.

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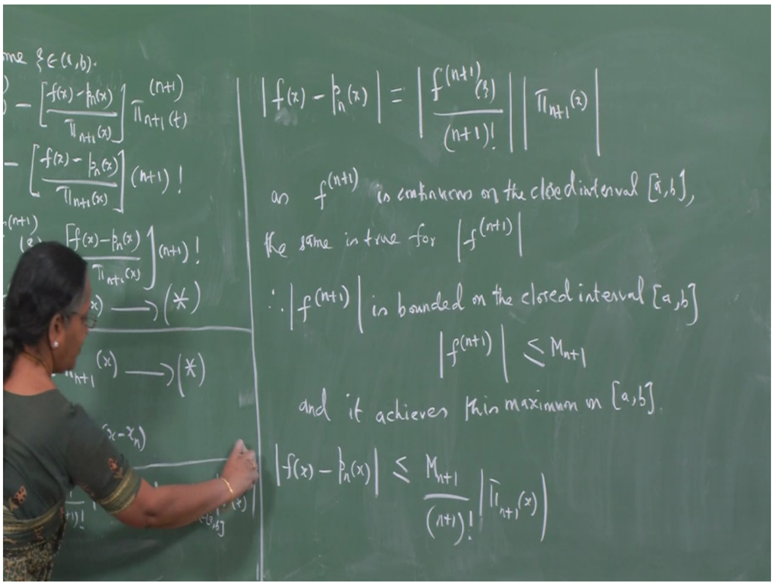

I consider modulus of F(x) + P n of x, so that will be modulus of f(x) + P n of x so that will be modulus of n + 1 derivative of Xi by n + 1 factorial into modulus of Pi n + 1 of x. As the n + 1 derivative to is continuous on the closed interval a, b, the same is true for absolute value of the n + 1 derivative of f on the interval a, b. And therefore modulus of the n + 1 derivative of f is bounded on the closed interval a, b. So modulus of the n + 1 derivative of f is less than or = M n + 1 and it achieves this maximum on the interval a, b and so we have modulus of f(x) + P n of x is less than or = M n + 1 by n + 1 factorial into modulus of Pi n + 1 of x and that is what we have to prove and which is given here.

So we have been able to show the error in interpolation and also the, an estimate of the size of the bound on the error in interpolation namely this result. So with the help of this inequality we can provide an estimate on the size of error bound when we interpolate a given function f(x) by means of an interpolating polynomial of degree at most n. We will look into these details in the next class.