Numerical Analysis

Professor R Usha
Department of Mathematics
Indian Institute of Technology Madras

Lecture - 7

Divided Difference Interpolation Polynomial

In the previous classes we learnt to obtain interpolating polynomials using forward differences and backward differences in the case when the information about the function is given at a set of nodal points which are equally spaced.

Then we learnt to construct an interpolation polynomial known as Lagrange interpolation polynomial for the case with the information about the function you specify that a set of points x i which may be arbitrarily located. So now in this lecture we shall see how we can obtain an interpolating polynomial using divided differences.

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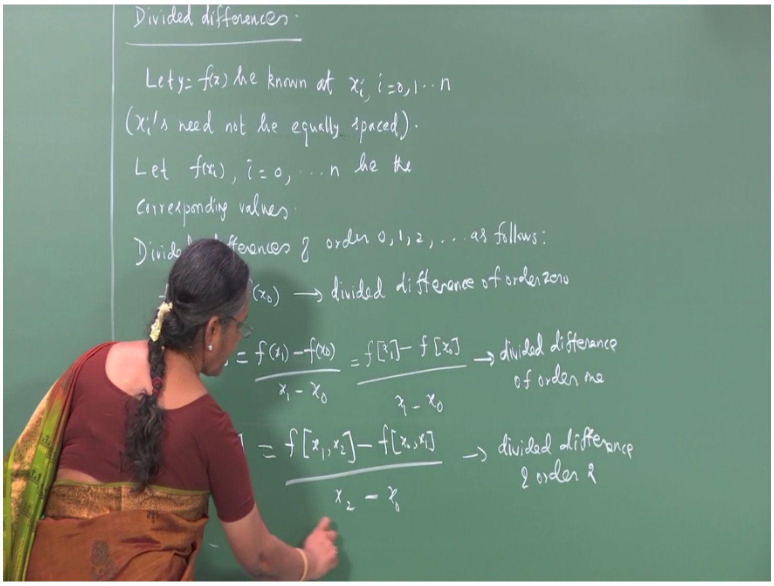

So this requires definition of divided differences of various orders, so we first introduce the divided differences. So let y equal to f(x) be known at say points x i for i is equal to 0, 1, 2, 3 upto n x i need not be equally spaced and let f(x i) for i is equal 0, 1, 2, 3 upto n be the corresponding values. So we define divided differences of orders 0, 1, 2, 3 etc as follows.

So I denote by the zeroth order divided difference as f(x 0) and that is nothing but the value of the function at x 0. And then this is a divided difference of order 0. And then I introduce a divided difference of order one and denote it by f(x 0) X1 which is f(x 1) minus f(x 0) divided by X 1 minus X 0. The notation is f(x 0) X1 is f at X 1 minus f at x 0 by X 1 minus X 0. But you know that f(x 1) is the zeroth order divided difference at X 1 minus zeroth order divided difference at x 0 by X 1 minus X 0.

So the first order difference divided difference can be given in terms of the zeroth order divided differences. So this is the definition for the divided difference of order one. So we write down now the divided difference of order 2 and denote it by f(x 0 X 1 X 2).

So that will be f(x 1 X 2) minus f(x 0 X 1) divided by X2 minus x 0) and we have just now constructed the divided difference of order one. So we use that definition here and divided difference of order one we again use the definition and then write down what the divided difference 2 is. So this is divided difference of order 2.

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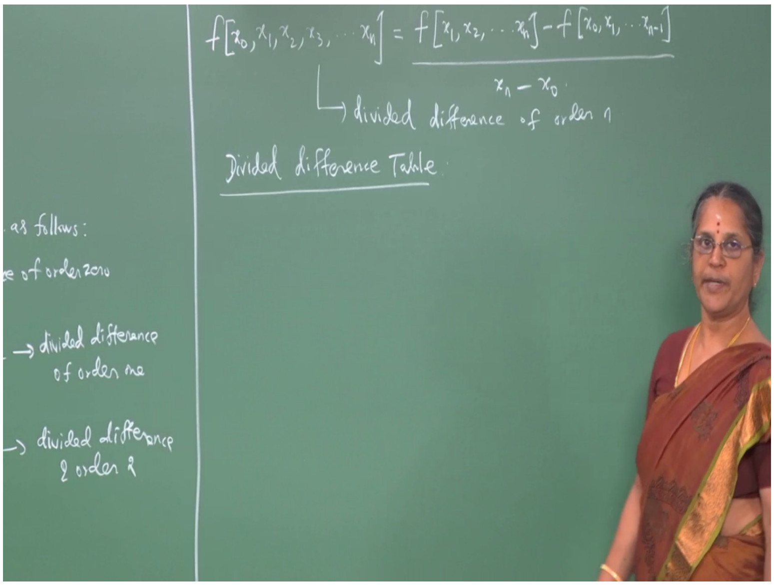

So we continue this way and I down this divided difference of order n as f[x, x 1, X 2, etc x n] minus f[x 0, x 1, etc upto x n minus 1] divided by it is going to be now Xn minus X 0. So this is divided difference of order n. So it is possible using this definition to construct the divided difference table and from there we will be able to write down the interpolating polynomial using the information from this table.

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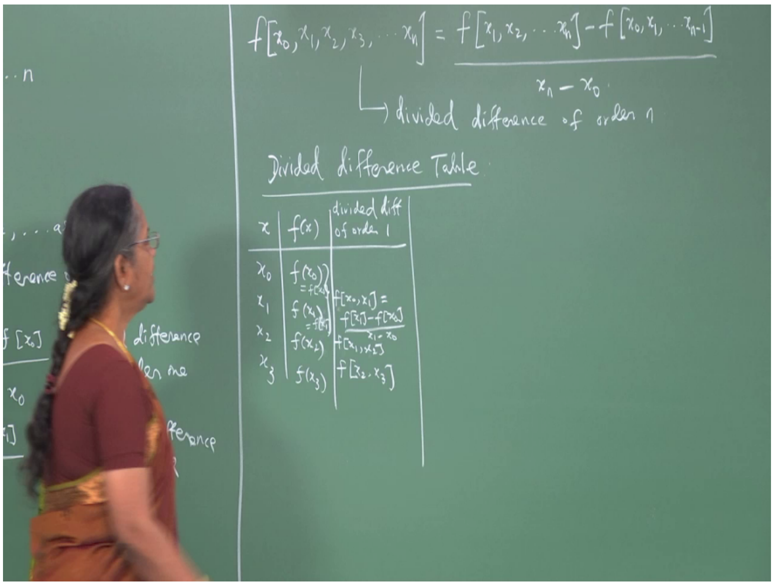

So we are given information about x and f(x) at a set of points say X 0 X1 X2 X3 and the corresponding values are f(x 0)f(x 1)f(x 2)f(x 3). So I would like to form the divided difference table for the given information. So I write down the column in which I'm going to get the divided differences order one.

So by definition we know this is the divided difference of order 0. This is the divided difference of order zero and so on so I require the entries in divided difference of order 1. So I know it is f(x 1) minus f(x 0) by X 1 minus X 0 and the notation is f(x 0 X1 and by definition it is f(x 1) minus f(x 0) divided by X 1 minus X 0.

So I write this entry between these two entries and denote it as f(x 0 x 1). So if I use the information here the next entry must be f(x 1 x 2). And I know the definition f(x 2) minus f(x 1) by X2 minus X 1. And then the entry between these two will give me a divided difference of order one and that is f of x 2 x 3.

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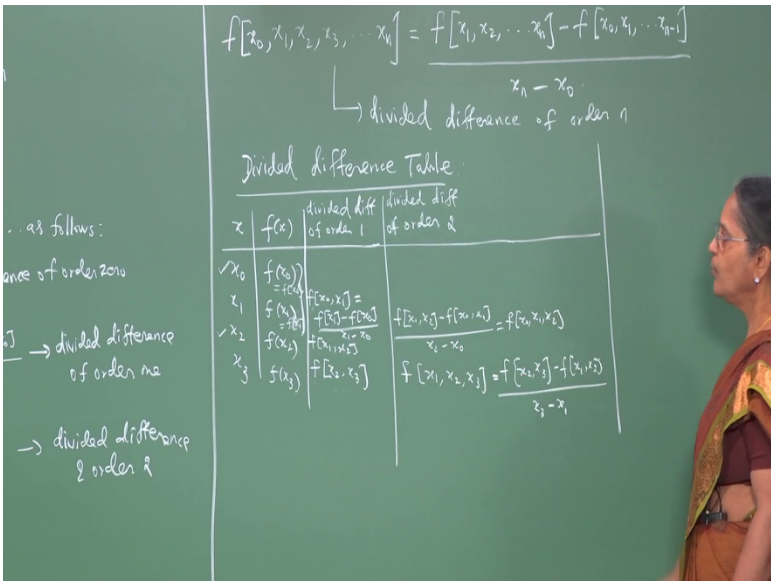

So having got the divided differences of order one from this table we move on to the divided differences of order to order 2. So again if you look at the definition it is f of(X 1 X 2) minus f(x 0 x 1) by X 2 minus X 0. So I have these two entries here so write in the middle write f of(X 1 X 2) minus f(x 0 x 1) divided by remember it should be in out x2 minus x 0 and this is what you denote by f(x 0 x 1 X 2). So similarly you have these two entries and so you will get the second order divided difference which is f of(X 1 X 2 x 3) what is it it is f of(X 2 X 3) minus f(x 1 x 2) divided by x 3 minus X 1. So this will be f(x 2 x 3) minus f(x 1 X 2) divided by you have used information at points X1 X2 X3 so it is divided by X3 minus x 1.

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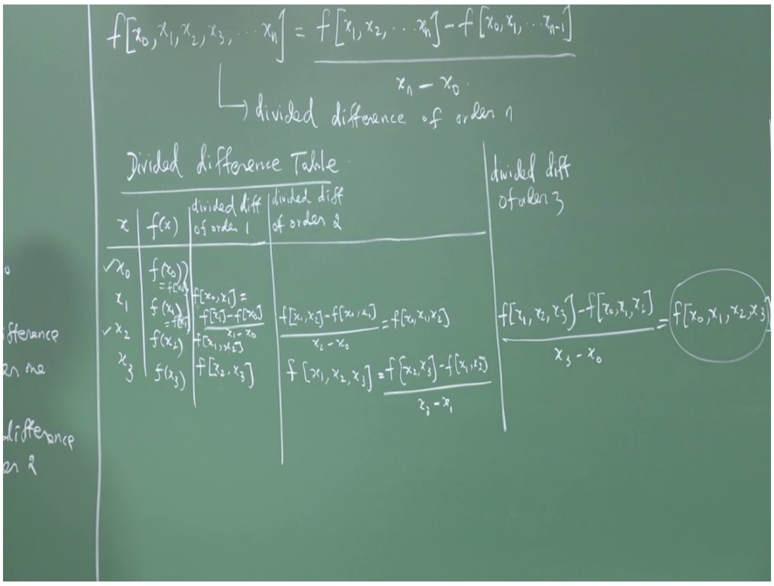

So you have constructed the column in which divided differences of order 2 have been obtained. So let us now work out the column in which we have divided differences of order 3. So we have divided differences of order 2 and hence if we find f (x 1 x 2 x 3) minus f(x 0 x 1 x 2) and divide, so this minus this will give you; so what is it that we have to write down x 3 minus x 0 and that is going to before (x 0 x 1 x 2 and x 3). So this gives you the divided difference of order 3 and this is constant. So the higher order divided differences beyond the third order divided difference will all be zero for the information given in this table. And the third order divided difference is constant. So given some information about f at discrete points we use the definition of divided differences and have found the divided difference table. So with this information we should see now whether we can construct an interpolating polynomial that interpolates a function f(x) at effects of discrete points x i where the x i need not be equally spaced.

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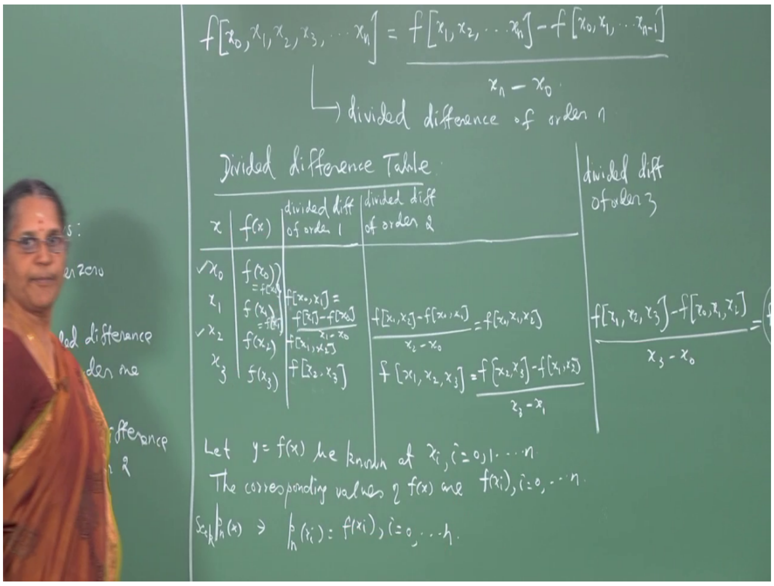

So let us assume that we are given information at n plus 1 points. So let Y equal to f(x) be known at x i for i is equal to 0,1,2,3 upto n. The corresponding values of f(x) are f(x i) at equal to 0,1,2,3 upto n. So I want to reconstruct this function f (x), so I seek a polynomial p n(x) of degree at most n that interpolates this function f(x) such that, so seek p n(x) such that p n (x i) is f(x i) for i is equal to 0,1,2,3 upto n.

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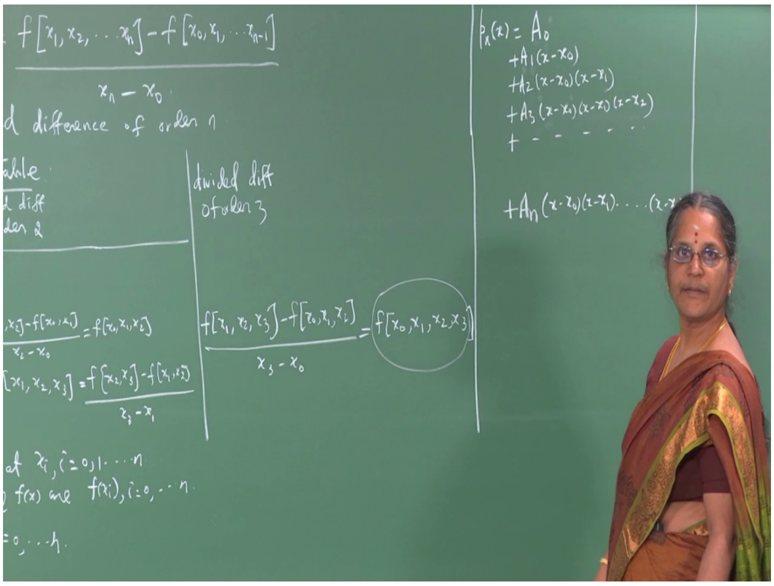

So in order to do that let us write a polynomial p n(x) in this form namely let it be A 0 plus A 1 int x minus x 0 plus A 2 into (x minus x 0) into (x minus x 1) plus A 3 into (x minus x 0 x minus x 1) into (x minus x 2)plus etc. Now you know the pattern in which every term is added to this polynomial. so I see that the first term is the constant the second term is a linear polynomial the third term is a second degree polynomial the 4th term is a third degree polynomial and so the n plus 1 th term will be a n into an n th degree polynomial x minus x 0 etc upto x minus x n.

So when it is the second term the coefficient is 1 first term coefficient is a suffix 0 third term coefficient is a suffix 2 and so on. So this is the n plus 1 th term so the coefficient is A suffix n. So we seek a polynomial in this form so let p n(x) be this. So what do we want we want p n to satisfy condition p n(x i) is f(x i).

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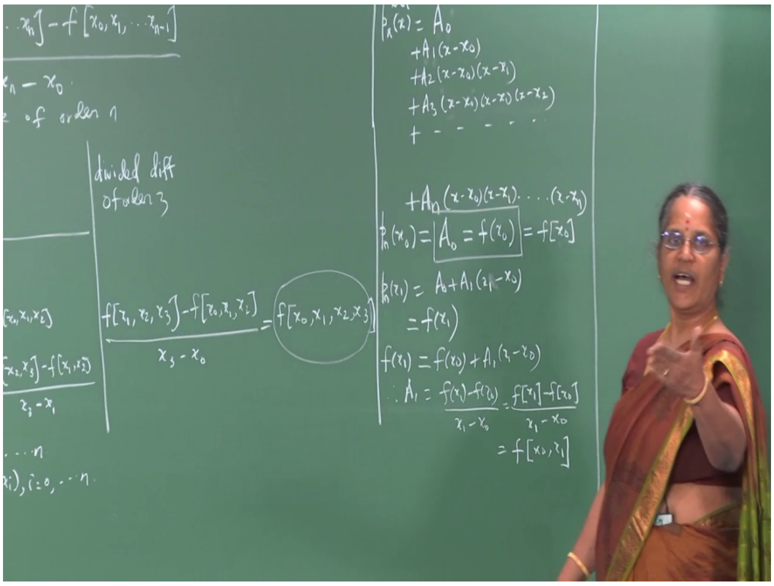

So let us find out find out what this condition is? So what is p n(x 0) it is simply A 0 because all these terms have x minus x 0 as a Factor in them. So what is a 0 p n (x 0) must be f(x 0) so A 0 is obtained but what is this is the zero th order divided difference f(x 0). So we apply condition p n(x 1)is going to be A 0 plus A 1 into X 1 minus x 0. Write A 0 but what is p n(x 1) it is f(x 1).

So this tells you that f (x 1) is equal to A 0 which is f(x 0) plus A 1 into x 1 minus x 0. Therefore A 1 can be written as a f (x 1) minus f(x 0) by x 1 minus x 0. But that is not but zero th order difference f(x 1) minus zero th order difference f (x 0) by x 1 minus x 0 and this is by definition the first order derivative difference f(x 0 x 1). So we saw that A 0 is the zero th order difference of f (x 0). A 1 is the first order difference f(x 0 x 1). Let us workout the coefficient A 2 and then we can generalize and write down what A n is in this polynomial.

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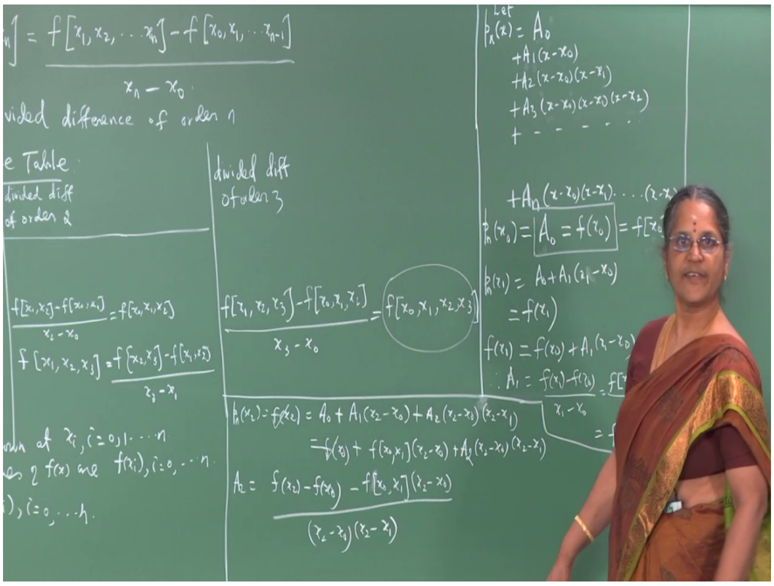

So let us find out what A 2 is? So I satisfy the condition p n(x 2) must be equal to f(x2). So P n(x 2) is f(x 2) and that is equal to so we use that A 0 plus A 1 into x 2 minus x 0 plus A 2 into x 2 minus x 0 into x 2 minus x 1. So we substitute for A 0, A 0 is f (x 0) plus A 1 we have already obtained A 1 and what is it? ummm and that is f(x 0 x 1) multiplied by x 2 minus x 0 we do not know what A 2 is and we are trying to determine A 2 using this condition. So from here we obtain A 2 as f (x 2) minus f(x 0) so this minus this minus f(x 0 x 1 multiplied by (x2 minus x 0) divided by (x2 minus x 0) into x 2 minus x 1).

So A 2 is given by this expression and we expect from the pattern of the coefficients that we have got, since A 0 is the 0 th order divided difference A 1 is the first order divided difference, we expect that this should give us the second order divided difference. So let us do manipulation on the terms that appear here and try to see whether what we think is going to be correct.

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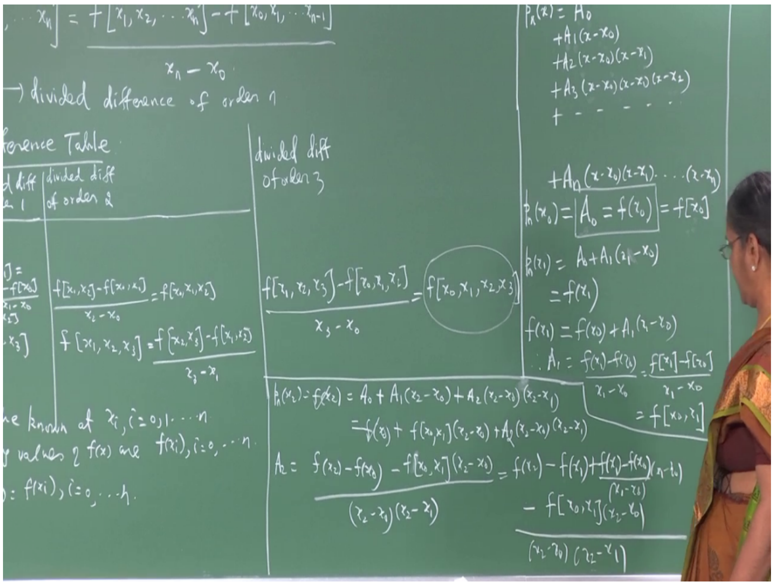

So we rewrite this term as follows, this is f (X 2) minus f(x 1)plus f(x 1) minus f(x 0). So I have subtracted f (x 1) and added f(x 1) then minus. What I do is I shall take this part I divide by X 1 minus x 0 and multiply by x 1 minus X 0. So I have not done anything except that I have added the term f(x1). The other terms remain as they are, So then I have minus f(x 0 x 1) multiplied by (x 2 minus x 0) that is this term the whole thing is divided by (x 2 minus x 0) into (x 2 minus X 1). So let us try to see what this gives us.

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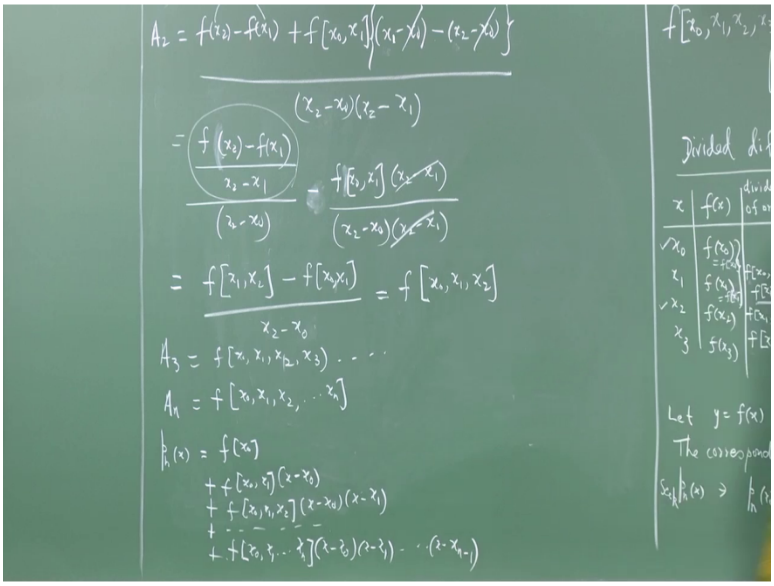

So A 2 will be equal to the first two terms let's write them as they are f(x 2) minus f(x 1). Let us look at the next term it is f(x 1) minus f(X 0) by x 1 minus x 0, just see this which can be expressed in terms of zeroth order divided difference and is the first order divided difference f(x 0 x 1), so this term is the first order divided difference f(x 0 x 1), so we write down that term as plus f[x 0, x1] but that is multiplied by x 1 minus x 0. So that into x 1 minus x 0. Then I have the next term which also has the first order difference f(x 0 X 1) and that is multiplied by (X 2 minus X 0) with a negative sign.

So I shall take this term so this will be minus x 2 minus X 0. So we have taken into account the numerator, the denominator is whole thing by x 2 minus x 0 into (x 2 minus X 1). So we write this as f (x 2) minus f (x 1) by x 2 minus x 1 by X 2 minus x 0 . So I take these two terms by the denominator and put it in this way. So I am left with plus f (x 0 x 1) into so i have a minus x 0 plus x 0 and I am left with (x 1 minus x 2).

So I shall write that as minus x 2 minus x 1. I have x 1 minus x 2 I rewrite that as minus (x 2 minus x 1)by the denominator (x 2 minus x 0) into (x 2 minus x 1). So cancelling they are distinct numbers so I am left with this so can you tell me what this is ? This is the 0 th order divided difference at x 2 0 th order divided difference at x 1 by x 2 minus x 1 and therefore this must be the first order divided difference f(x 1 x 2)and what do we have here? It is the first order divided difference (x 0 x 1) and both the terms are divided by x 2 minus x 0.

So recall the definition of the second order divided difference it is f(x 1 x 2) minus f(x 0 x 1) by (x 2 minus x 0) and we denote this by f[x 0 x 1 x 2] which is the second order divided difference. By looking at A 0 and A 1 we show that yes it is so. So following in the same way we can show that A 3 will be the third order difference [x0, x 1, x 2, x 3] and if I continue this way one show that A n is the n th order divided difference x 0, x 1, x 2 etc x n.

So which means that all the constants which appear in our polynomial p n(x) have been computed in terms of the various order divided differences. So we substitute so we obtain therefore the interpolating polynomial p n(x) of degree at most n that interpolates the function f(x) at a set of discrete points x i for i is equal to 0 to n and it is given by A 0 which is f(x 0) plus A 1 which is f(x 0, x 1) multiplied by x minus x 0 plus A 2 which is f(x 0 x 1 x 2) multiplied by x minus x 0 into x minus x 1 plus etc. So I write down f An which is x 0 x 1 etc x n into the factors will be x minus x 0 x minus x 1 etc upto x minus x n minus 1.

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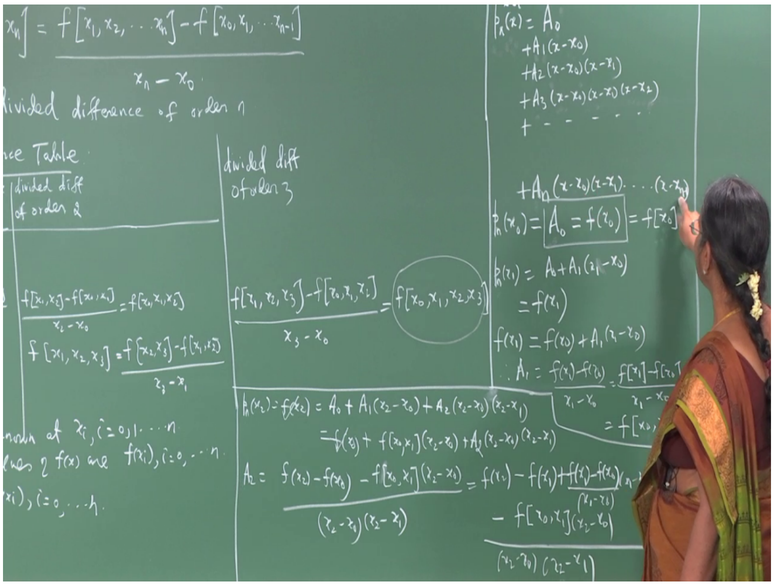

So here this is x n minus 1if it is A 3 we start with two its a polynomial of degree 3 when it is A n then we have a polynomial of degree n it starts at x 0 so it goes upto x n minus 1. So there are n such factors so the degree of this polynomial will be n.

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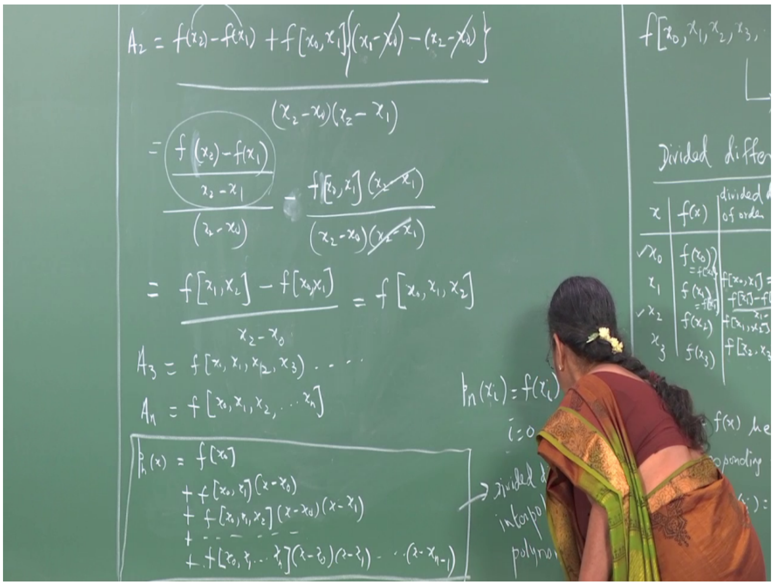

So we have constructed a polynomial p n(x) of degree at most n with the property that p n(x i) will be equal to f (x i) for i is equal to 0 1 2 3 upto n. And this polynomial is called the divided difference interpolating polynomial divided difference interpolating polynomial.

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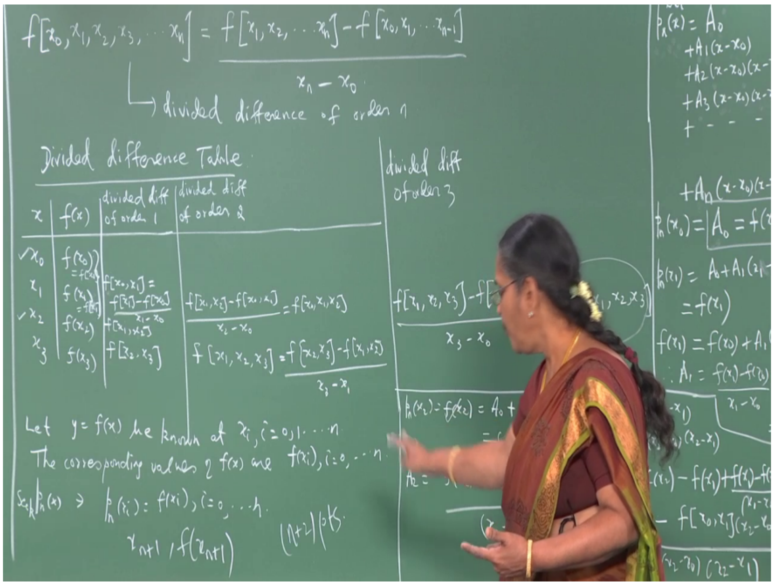

The greatest advantage of divided difference method is the following if suppose I have another information about this function f(x) write about this function f(x) say at the point x n plus 1 and the information is f(x n plus 1) its just not at 1 point whatever be the number of points at which you get extra information about a function the following can be done we just explain this in case we have one additional information.

If we have one additional information then we have totally a set of n plus 2 points at which the information is available and therefore we want to reconstruct this function whose values are known at a set n plus 2 discrete points and therefore we can seek a polynomial of degree n plus 1 that interpolates this function at a set of these n plus 2 points.

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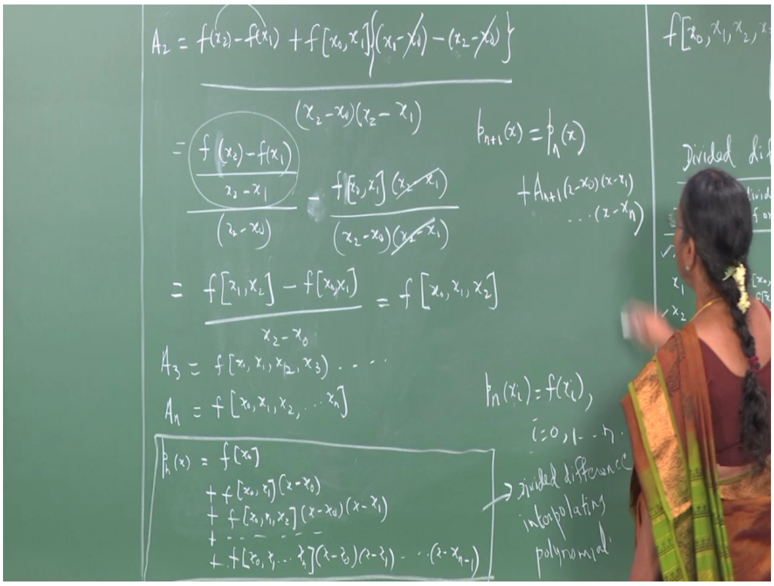

Already we have constructed a polynomial that interpolates the same function at a set of n plus 1 points and this has been obtained using divided differences. Alright! we now have an extra information so we do not want to waste any of the efforts that we have put in, in already computing an interpolating polynomial that interpolates at n plus 1 discrete points but some little efforts that we need to give so that we get now a polynomial of degree at most n plus 1 that interpolates at a set of n plus 2 points now using this extra information at x n plus 1 f(x n plus 1).

The procedure is very simple in the sense that I require the polynomial of degree n plus 1 and that is going to be the polynomial of degree n but I have already computed plus another term must be added to it so that it is A n plus 1 multiplied by x minus x 0 into x minus x 1 etc upto x minus x n. So you observe that there are n plus 1 such factors so this term is a polynomial of degree n plus 1.

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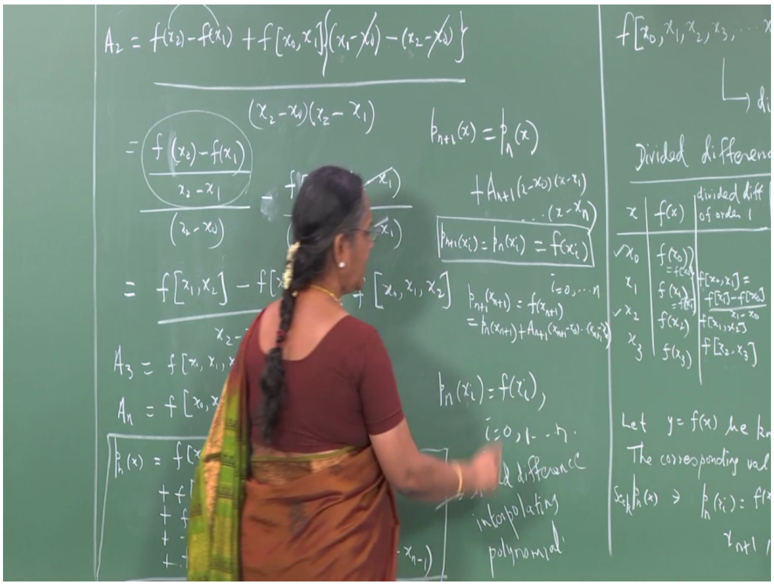

Now I would like to see what happens to this polynomial at the point i is equal to 0 1 2 3 upto n and I observe that it is equal to p n(x i) and because when i is equal to n this term x n minus x n will be 0 if i is 0 x 0 minus x 0 will be 0. So at all these points the contribution from this term will be 0. So you essentially have p n plus 1 (x i) is pn (x i) that is an interpolating polynomial interpolating the function f(x) at points x i for i is equal to 0 to n.

So p n plus 1 will satisfy the condition that at all the x i from 0 to n p n plus 1 (x i) is f(x i). So now let us see what will be the p nplus 1 at x n plus 1 at which the additional information is given. It is given that it must be equal to f(x n plus 1) and so that must be equal to p n evaluated at x n plus 1 which can be done plus A n plus into x n plus 1 minus x 0 etc upto x n plus 1 minus x n.

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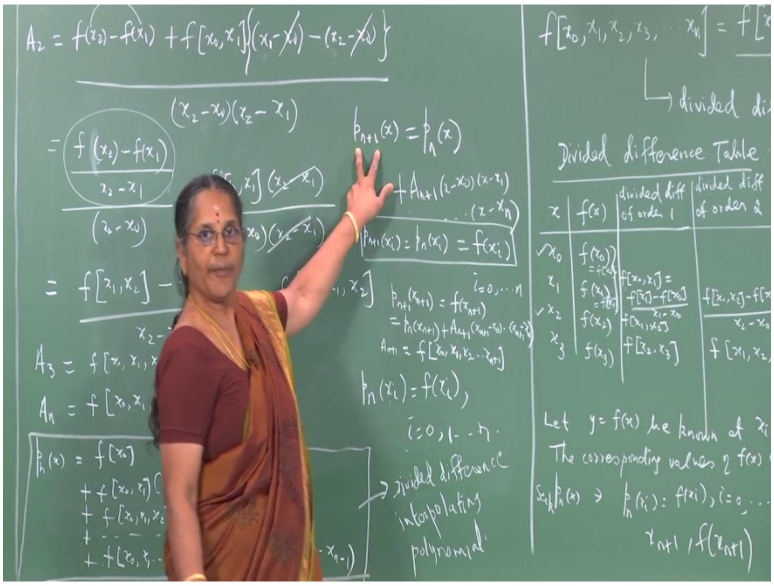

So from here we can determine A n plus 1, and we know from what we have already done we will end up with A n plus 1 to be the divided difference of order n plus 1 namely it'll be f(x 0 x 1 X 2 etc upto x n + 1) which we can compute. So the a n plus 1 is known we can feed it here and arrive at the interpolating polynomial of at most of degree at most n plus 1 that interpolates the function at a set of n plus 2 points.

So given the additional information about the function we do not have to waste our efforts which we have already put in to construct a polynomial of lower degree use it and add another term that incorporates this additional information given about the f and that term will give you a polynomial of the next degree finally ending up with the interpolating polynomial of the required degree that interpolates the function at a set of n plus 2 points.

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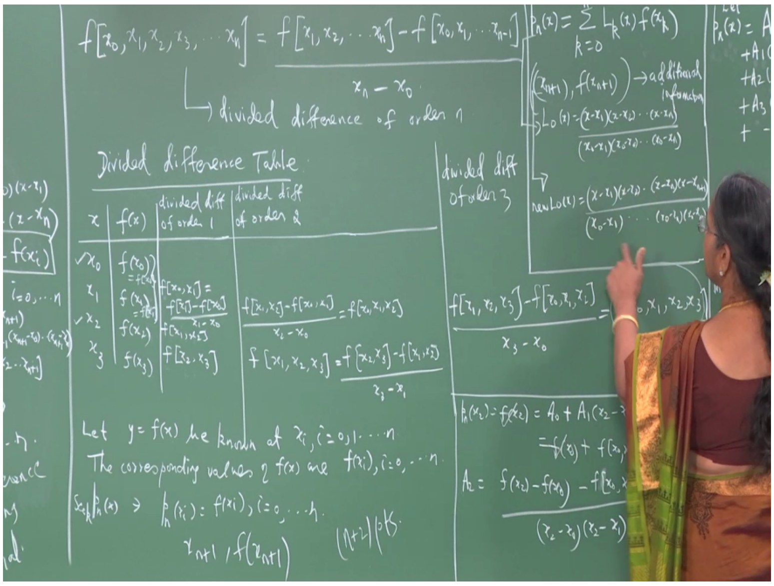

What is the greatest advantage of divided difference method and the amount of calculations that are needed to construct the divided difference table is very much reduced as compared to the construction of Lagrange interpolation polynomial. So why do you say that divided difference method scores over Lagrange interpolation polynomial?

The reason is this in case you have already computed p n(x) using Lagrange interpolation polynomial say given the information at the first set of n points then what is it? it is nothing but k is equal to 0 to n l k(x) equal to f(xk) where l k(x) can be written down. Now you are given information say at x n plus 1 and the information is f(X n plus 1)this is an additional information. Right? The question is can you do something like what you have done in the case of divided difference table.

No it is not possible because for example in constructing p n(x) what was your l 0 (x) it was x minus x 1 into x minus x 2 etc x minus x n divided by x 0 minus x 1 into x 0 minus x 2 etc upto x 0 minus x n. But if you want to construct a polynomial of degree now n plus 1 to incorporate this additional information the new l 0 (x) must be x minus x 1 into x minus x 2 etc upto x minus x n into x minus x n plus 1 divided by x 0 minus x 1 etc x 0 minus x n into x 0 minus x n plus 1. So this must be multiplied by f(x 0) to give you the first term.

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Similarly each of those polynomial l k (X) and p n plus 1(x) now will be Sigma K is equal to zero to n plus 1 L k(x) into f (x k) so each of these polynomials l k(x) where k equal to 0 to n plus 1 will have in both the numerator and the denominator extra factor that comes because of the information at x n plus 1, f (x n plus 1). And therefore the previous calculations are their parts that we have put in to construct a polynomial of degree n that is the function at a set of N + 1 points will not be of any help to us in getting a polynomial of degree n + 1.

And therefore the calculations have to be performed once again to construct this polynomial of degree n plus one where an additional information is given about the function. Whereas this did not happen in the case of the divided difference formula the information the polynomial that we already have obtained is going to be used and we just add an extra term that takes into account the information that is provided to us at this extra point. And that is why the divided difference method scores over Lagrange interpolation method.

So now that we know the definition of what the divided differences are? How to construct divided difference table and what is the interpolating polynomial that is based on divided differences. Let us take a simple example and then construct an interpolating polynomial based on divided differences.

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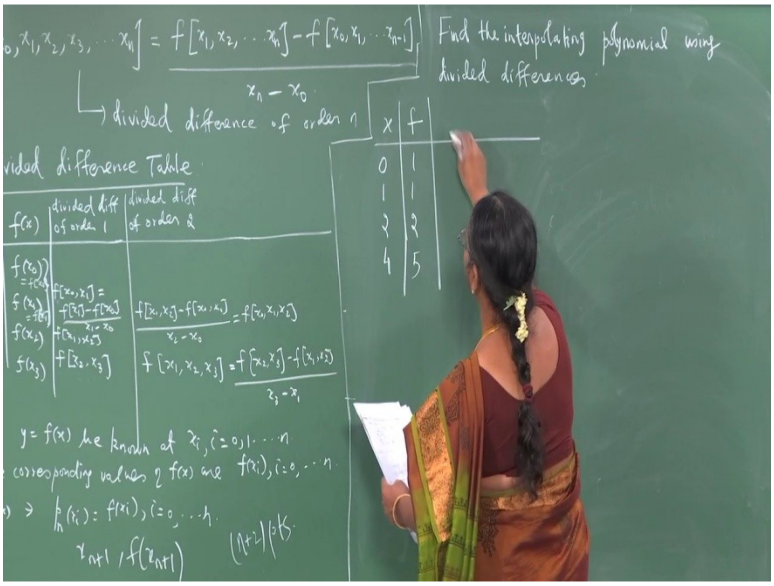

Ok Let us find the interpolating polynomial using divided differences. So the information is given at x and the corresponding f values 0 1 2 and 4, and the corresponding values are 1 2 and 5 I am just illustrating the procedure so taking some simple example where information at Four distinct Points are given. So the first thing that we should look at is; are the x values equally spaced? No they are not equally spaced.

So in order to get an interpolating polynomial I can use either Lagrange interpolation polynomial or divided difference interpolation polynomial. So now that we have learned divided differences let us use the divided differences and that is what the question is so we shall obtain the divided difference stable and then construct that polynomial.

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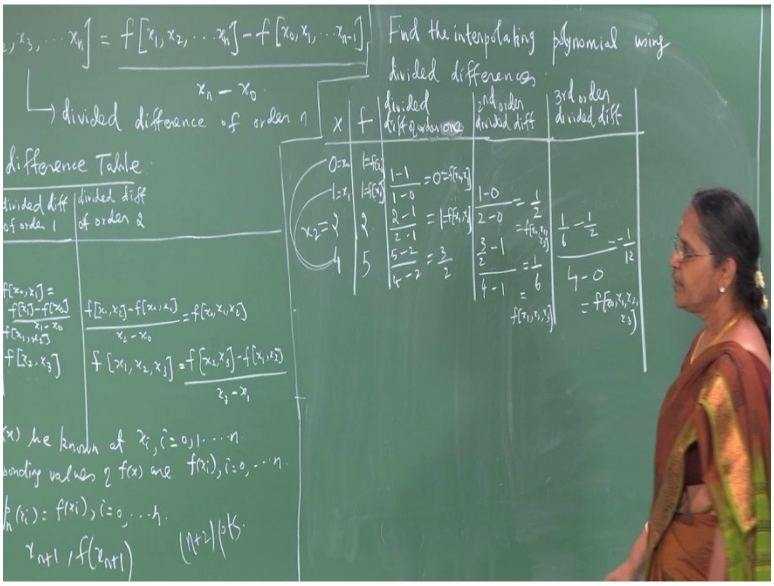

So I form the divided differences of order 0, so what is it? this is our x 0 this is x 1 so this is our zeroth order divided difference x 0 this is first zeroth order divided difference f(x 1). So the divided differences of order 1 can be constructed by this minus this so 1 minus 1 divided by x 1 minus x 0 so 0 then 2 minus 1 by 2 minus 1 then 5 minus 2 by 4 minus 2, so these are the divided differences of order 1 and we know that this by notation is f(x 0 x 1).

So now I compute second order divided differences. So this is by notation f(X 1 X 2). So how do you compute the second order difference. The difference between the successive first order difference is 1 minus 0 divided by x 2 minus x 0 this is x 2 and that is x 0. So 2 minus 0 so it is half. And then 3 by 2 minus 1 divided by X 2 minus X 1, so 4 minus 1, so it is going to be half by 3 so 1 by 6.Hhow do you denote? This is f(x 0 x 1 x 2) and this is f(x 1 x 2 x 3), these are the second order differences so we form the third order divided difference.

So that is going to be by definition 1 by 6 minus half divided by x 3 minus x 0 so it is 4 minus 0 and the value is going to be minus 1 by 12. So the third order given information at four distinct points the third order difference divided difference is constant and the divided differences of order beyond 3 will all be 0. And how do you denote this, this is f(x 0 x 1 X 2 x 3)

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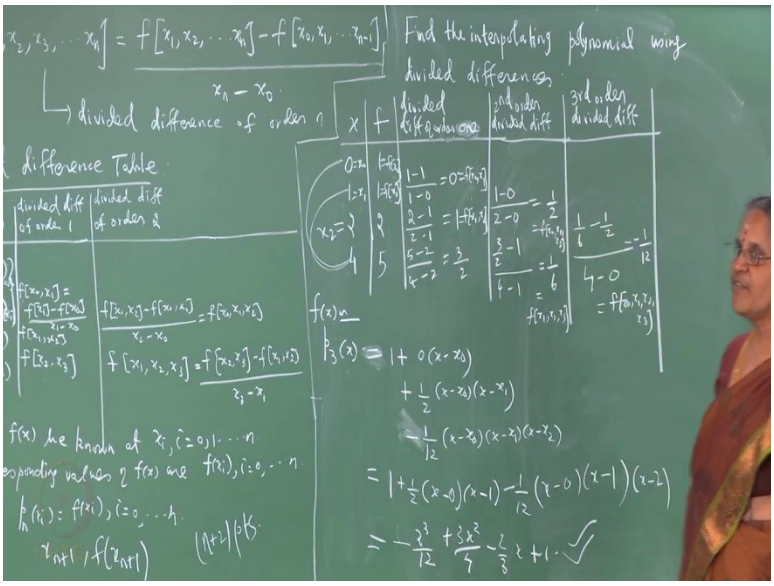

So we use this information in seeking a polynomial of degree 3, so the required polynomial p 3(x) that approximates the function f (X) is given by first term f(x 0) which is 1 plus f(x 0 x 1) 0 into x minus x 0 plus f(x0 X 1 X 2) half into X minus x 0 into x minus x 1 plus f(X) X1 X2 X3) minus 1 by 12 into x minus X 0 into x minus x 1 into x minus x 2. So the polynomial is 1plus half into (x minus x 0 is 0) into x minus x 1 is 1) minus 1 by 12 into (x minus x 0 0 x 1 is 1 and x 2 is 2).

So you see that you have a third degree polynomial that approximates this function or reconstructs these function errands in this interval between 0 and 4. So if you simplify this it turns out to be minus X cubed by 12 plus 3 x square by 4 minus 2 by 3x plus 1. So this is required interpolating polynomial of degree 3 that interpolates the function at a set of four distinct points. So we now have understood how we can construct interpolating polynomial using divided difference.