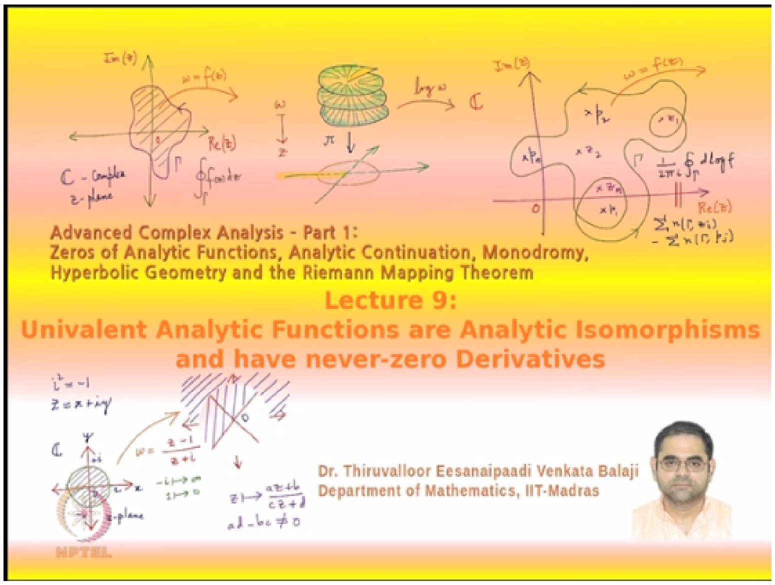
Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

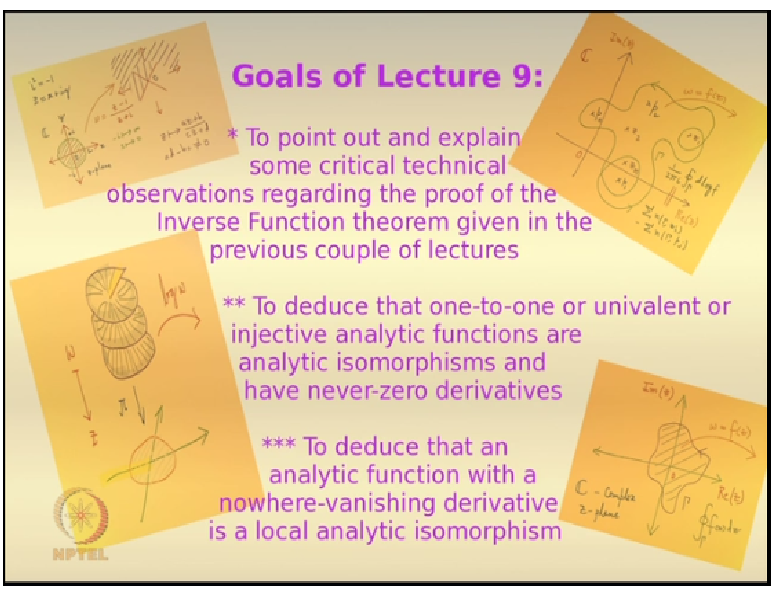
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-09
Univalent Analytic Functions have never-zero Derivatives and are Analytic Isomorphisms

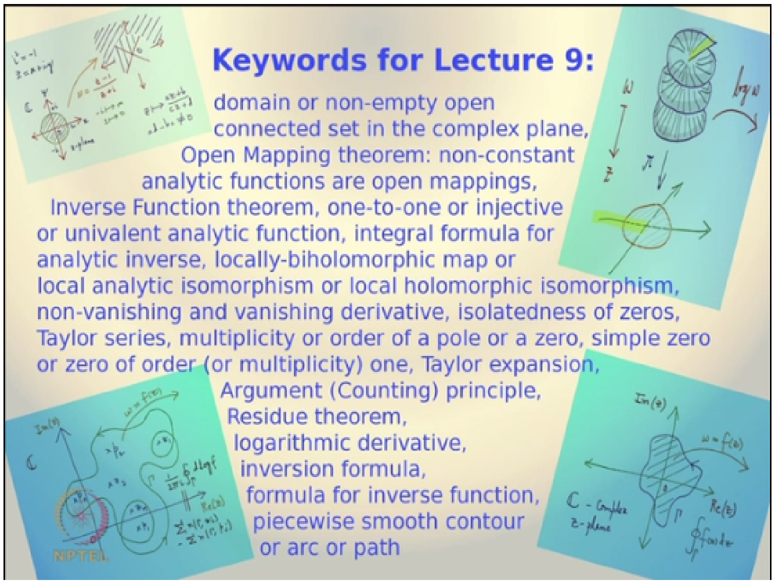
(Refer Slide Time: 00:05)



(Refer Slide Time: 00:09)

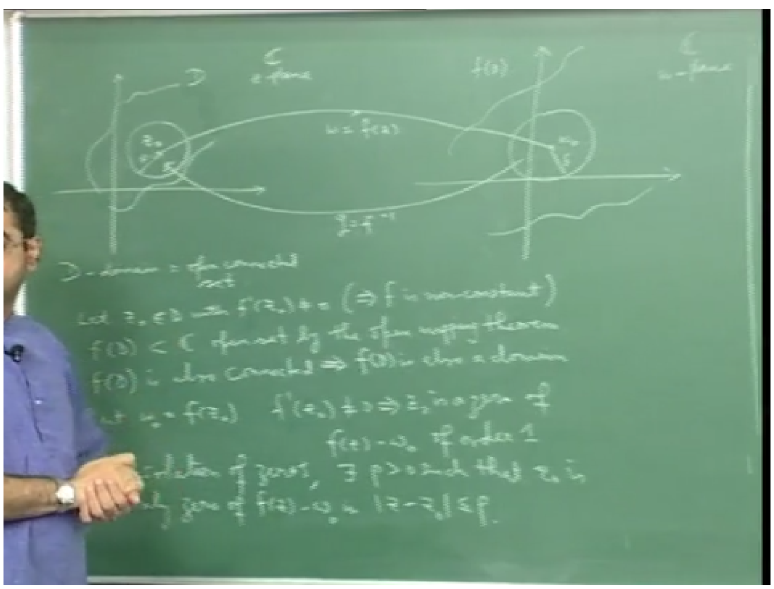


(Refer Slide Time: 00:18)



Okay, so let us continue with our discussion of the inverse function theorem okay, so I want to point out certain technicalities with respect to inverse function the proof that I gave in the previous lecture. So, let me again look at the proof okay.

(Refer Slide Time: 00:53)



So, you have certain domain D the complex plane well it need not be bounded is some domain D the complex plane which is the z plane and you have function w equal to f of z defined on this domain. So, this D is the domain which is an open connected set okay. And of course the open set is non-empty and you suppose we take a point z0 in the domain such that f dash the derivative of f at z0 does not vanish okay.

Let z0 be a point in the domain with f dash of z0 not equal to 0 okay then of course if you look at the target complex plane. This is source plane the source variable is z that target variable is w this is the w plane and w is a f of z and of course image of D will be f of D okay by the open mapping theorem f of d will also be an open connected set see the first thing you should this.

If the derivative first of all the derivative is not 0 so, it is not it is a non-constant analytic function. Because if it is a constant analytic function the derivative will be identically 0 so, it is a non-constant analytic function first. And the other important thing is that since this domain is actually connected it cannot be constant on some disc okay.

It cannot just be constant on a small disc right because identity theorem will tell you that I fit is constant on a small disc then it is constant everywhere right that is because the domain is connected okay therefore this function is a non-constant analytic function disc because of this continuation alright and the open mapping theorem says that the non-constant analytic function maps open sets to open sets.

And since D is an open set f of D will also be an open set so, you see this f of D will be an open set in the complex plane by so, by the open mapping theorem and see the point I want to make is that so, **so** let me write it here this implies f is non-constant. So, f of D is an open set and you know under a continuous function the image of a connected set is connected since D is connected f of D is also connected.

So, f of D is also connected and this implies that f of D is also a domain there is also a domain alright. So, f of D I mean so, D for me is some region with some which maybe unbounded it may have partial boundary okay. And then I will have similarly I will have some f of D it will be some region with some boundary here. And it may not be fully bounded in all directions okay.

But is an open connected set okay and you take let the value f take the value w0 at f of z0. So, put w0 equal to f of z0 okay now see what f dash of z0 is not equal to 0 tells you is that z0 is a 0 of f of z-w0 of order1 okay f dash of z0 is non-zero implies z0 is a 0 of f of z-w0 of order1 this is what it means alright. Because if you have 0 of higher order.

Then all derivatives up to that all derivatives up to one less than that all will vanish okay which is 0 of order1 then the first derivative will not vanish. But the zeroth derivative which is a function which will that will vanish. If it is 0 of order 2 then the function will vanish at that point and the first derivative will vanish which is 0 of order 3 then the function it is first derivative and the second derivative will vanish.

The third derivative will not vanish okay so, all derivatives up to one less than the order will vanish okay. So, the fact that f dash does not vanish means that z0 is a 0 of order1 okay this is essentially if you want this follows from looking at the Taylor expansion or by looking at the definition of what is meant by order of a 0 okay Taylor expansion of f in a disc surrounding in a disc centred at z0 okay.

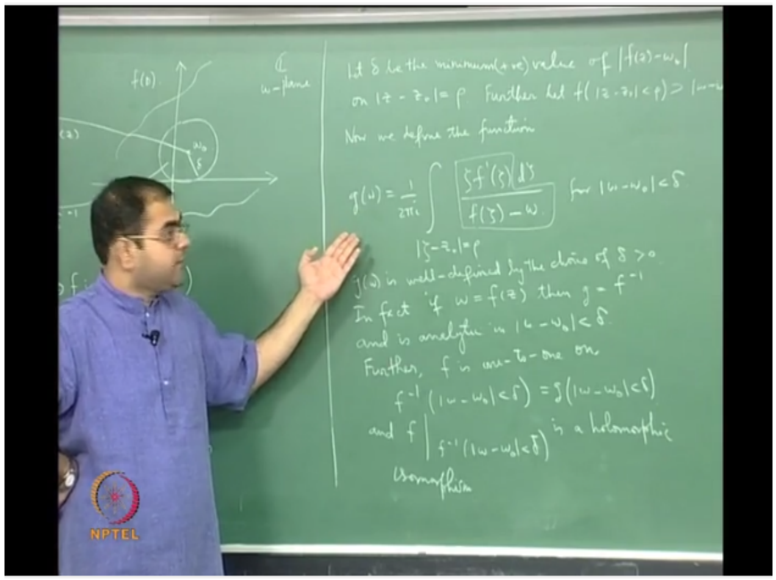
Now the point is that you look at the function so, you know you choose now since f is a non-constant analytic function f of z-w0 is also non-analytic function. And you know the zeroes of a non-constant analytic functions are the zeroes are isolated okay. Therefore what does it mean give me a 0 then there is a disc I can find disc okay closed if you want I can even include the boundary of the disc.

And make sure that there is no other 0 of that function except the point at the centre of the disc which is the chosen 0 okay. So, isolation of zeroes tells me that I can find a disc of radius row centred at z0 such that on this disc including the boundary there are no zeroes of f **f** of z-w0 except 0 at the centre z0 which is a 0 of order1 okay. So, by isolation of zeroes there exist a row positive such that z0 is the only 0 of f of z-w0 in mod z-z0 lesser than or equal to row okay.

In other words the function f of z assumes the value w0 only once in that closed disc and that is at the point z equal to z0 that is the centre of the disc. And the multiplicities only one it assumes the value only once because if it assumes the value twice then the order of the 0 of f of z-w0 at z0 will become 2 and so on. But the order of 0 is 1 so, it assumes the only 0 it has is at z equal to z0.

And the multiplicity of the 0 is 1 that is what you must understand okay now what you do is that you look at the function you look at the modulus of this function mod of f of z-w0. Now that modulus on this boundary is non-zero and it has a minimum okay.

(Refer Slide Time: 09:39)



And that is what we called as delta alright see let delta the minimum positive value of mod of f of z-w0 in on the boundary disc on the boundary circle okay see if you look at mod of f of z-w0 that vanishes only at one point in the disc that it is at the centre okay at no other point it vanish because if it vanishes at some other point that will become a 0 of f of z-w0 but we have assume that there is only one 0.

Because of isolation of zeroes so, in particular on the boundary circle also it will not vanish because it is modulus of a it will be a it is non-negative function okay. In fact it is a positive real valuehood function. And it is a continuous function when you restrict with to this set this is a compact connected set therefore the image of that continuous function on the real line will be a closed interval okay.

And this delta will be the left end point of that closed interval okay a continuous the I am just using topological facts that the continuous image of a connected set is connected the continuous image of a compact set is compact is subset of the real line is connected if and only if it is an interval subset of the real line is compact if and only if it is both closed and bounded and in interval if it is a close set.

Then it has to be a closed interval okay so, well all that gives you the delta is the there is there is this delta at p want n is positive okay. Now what you do is now we define the function g of w to be equal to 1 by 2 pie i integral over mod zeta-z0=rho zeta f dash of zeta d zeta by f zeta-w for mod w-w0 less than delta now you define this function okay.

Now so look at what is happening see mod w-w0 is strictly less than delta is the disc centred at w0 and radius delta in the target plane target complex plane the diagram I have already the way I have drawn the diagram it is shows that the disc is already in the image okay, the way I have written it it is shows that the disc is already in the image okay. but that is actually in the conclusion alright see you define this function first okay you define this function.

And first of all to say that this function is well defined you see what is the integrant, the integrant of this function is this function okay and where is this function considered it is considered on the boundary circle alright mind you I am integrating with respect to zeta and w is not part of the variable of the integration. So, after I integrate it I will get something that depends on w and that is what I am calling as g of w okay.

Now if you look at the integrant okay then you can see the integrant is continuous on this boundary circle. Because you see f is analytic therefore f dash is also analytic, so f dash is continuous and this is just z f dash z by f of z-w this is what that integrant is as a function of z where z varies on this boundary circle okay, I am only changing the variable of integration from z to zeta for an obvious reason which will soon become apparent okay.

But the fact is that this is integrant is continuous mind you this the only problem with the integrant the continuity of the integrant is that the denominator can vanish okay. But the denominator cannot vanish because is the denominator vanishes then I will get a value on the boundary circle for which f of I will get a z if I get a zeta on the boundary circle such that f of zeta=w okay.

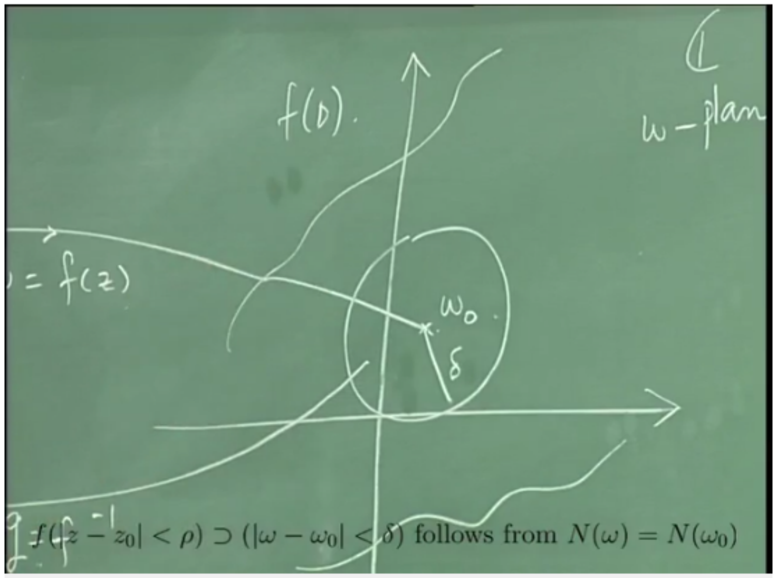
Then it will mean that the distance of f of zeta from w0 is less than delta but you know for every point in the boundary circle the distance of f of zeta-w0 is greater than or equal to delta okay which is a contradiction. Therefore this the denominator is never going to vanish therefore this is a continuous function it is a continuous function on the boundary therefore I can integrate and if I integrate I will get a function.

So, this function is well defined there is no problem about it okay to integrate a function I all I need is a continuous function on the curve the moment you have continuous function on a smooth or piece wise smooth curve okay you can always integrate it. And is the resulting integral is well defined is just the Riemann integral alright. So, these integral exist but it will depend on w because is w is here on the right side okay.

After you integrate it I will get something that depends on w and that is what I am calling as g of w, so this g of w is well defined there is no doubt about it alright and mind you g of w is a map defined on this disc and it is taking complex values. The fact is that this g of w will actually go back into this disc and it will actually be equal to z where f of z is w, so the fact is g of w is well defined by the choice of delta okay.

But the fact is g is nothing but f inverse the fact is that this g is the formula for f inverse and f and that g which is equal to f inverse is an analytic for of w in this disc. So, the picture is that you are going back, so your g is like this, so g=f inverse is defined like this okay. and see the fact the point is if you take the inverse image of if you take the inverse image of this disc under f okay. If you take the inverse of this disc under f, so maybe I will be a little careful and say that this disc I choose also delta small enough, so that this disc is inside f of D okay.

(Refer Slide Time: 17:09)



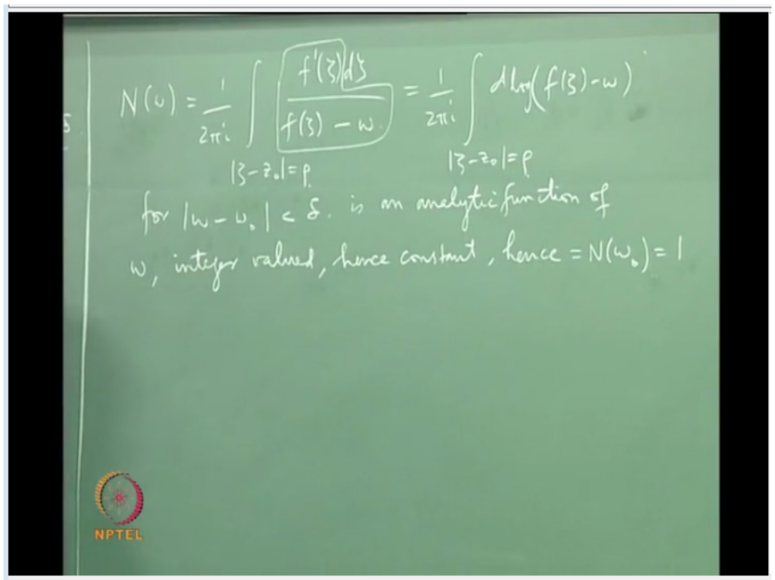
So, I perhaps it is not required but let me further let me write this further let f of the let the image of the disc mod z-z0 strictly less than rho contain the disc mod w-w0 less than delta let me assume this okay. so, I mean w0 is in the image alright and I have chosen a delta but if necessary let me make the disc delta smaller, so that I am I assure that this whole disc is in f of D okay let me assume that.

So, the fact that f the disc in f of D will tell you that every point in the disc is a value of f, so there is always a z such that f of z=w alright and the fact is that g what is this g of w it is f inverse of w and that is the same as z. So, this if you compute this you will get z okay where z is the where is the point at which f takes the value alright and more importantly you were also have that the function will turn out to be injective on the inverse image of that disc okay which is the either you take the inverse image of that disc under f or you take the image of that disc under g.

They are 1 in the same okay so, further so, what is happening is further what will you have if f is one to one on f inverse of that disc okay which is the same as g of that disc and mind you see f inverse is analytic f inverse is just g which is analytic. So, it is also an open map so, the image of this open disc will again be an open disc but it will be inside disc okay. So, what you must understand is that the inverse is defined okay.

And so, f will be one to one on this and f restricted to this disc f inverse of mod w-w0 less than delta is a holomorphic isomorphism okay. Now the fact the technical things that I want you to understand two points okay the first point is that f takes f is one to one on that disc on the inverse image of this disc. And the reason for f being one to one is because of this formula okay. And in fact you know why f is onto one on the disc is also.

(Refer Slide Time: 20:51)



Because of this function n of w that we defined see you can define the function n of w to be 1/2piei integral over mod zeta-z0 equal to row f dash zeta d zeta by f zeta-w okay. If you look at this what is this the right side is 1/2piei integral over mod zeta-z0 equal to row of d log f of zeta-w this is what it is this what the right side is and what is this what does this give you by the argument principle or the residue theorem what this tells you is you will get the number of zeroes-number of holes of f zeta-w in inside this disc okay.

But inside this disc this will how move poles because this integrand is this by the integrand is this by and this the integrand is continuous the denominator does not vanish. Because of the way you have chosen delta okay so, these the denominator does not vanish that means this has no poles alright. And it will have only one 0 and that see I mean it will have zeroes at the points zeta where f of zeta equal to w okay.

It will have zeroes those will you the points z which will be map to w and the number of times if you count those points with multiplicity that is an number you getting here. This is the number of times the function f assumes the value w inside that disc okay this is an integer alright But what we check was that this where is for what values of w it is defined it is for all those w with mod w-w0 strictly less than delta okay.

We check that this actually an analytic function of w right you know right uh going to back to the proof of the open mapping theorem alright. So, if this is an analytic function of w of w integer value hence constant hence it is also equal to N of w0 okay. You have an analytic function which is a integer value and is defined on a connected set. Therefore that integer value has to be only one value.

That means the analytic function is constant therefore it is the same value no matter what value for w I put in particular I can put w equal to w0. So, N of w is same as N of w0 but what does this but what is N of w0 N of w0 is 1 okay. In that disc the number of times f takes the value w0 is only one it takes the value w0 at the centre of the disc z0. And the multiplicity is 1 because this 0 is of order 1 for f of z-w0 at z equal to z0 okay.

Therefore what but so, what you get, you get for every w in this disc you get N of w is equal to 1 you get this fact okay now if you go back and if you think about this fact what this tells you it tells you two things. It tells you that for every w such that mod w-w0 is less than delta N of w equal to 1 means there is always as z inside this disc such that f of z is equal to w that is the first point.

And the second point is there is only one z and the multiplicity with which f assumes the value w at z is 1 because if z if it assumes the value w at z twice then this N of w will become 2. Because this N of w is with is number of times the function takes the value w. The function f of zeta f of z-w takes the value w I mean takes the value 0 which is same as number of times the function f of z takes the value w counted with multiplicity okay.

So, this is 1 so, what this tells you if you now recall this is not necessary this is automatic what it tells you is that the you need not assume this. So, the originally the diagram that I do was correct okay. This whole disc will automatically been the image this disc will be in the image in fact the disc will be not in f of not only in f of D. This disc will be in the image of this disc itself that is which is what I have assumed.

But it did not be assume it follows from here okay that is one technically point and what this will you, this will tell you that f restricted to the inverse image of that disc is one to one. Because what does that tell you it tells you every value w with mod w-w0 less than delta is assumed by f only once in the disc centred at z0 radius row okay excluding the boundary.

That means that is another of way of saying that f restricted to the inverse image of this disc is one to one okay that is what I have written here okay f is one to one on the inverse image of that disc okay that follows from there okay. And there is another techniquality that I want tell you see if you go back to the proof last time of proving that g of w is an analytic function of w at some point we needed that for w equal to f of z.

We need that f dash of z is non-zero okay we needed that the derivative does not vanish at z okay and I am saying now that is also a consequence of this okay see let me recall for a benefit how we got this g of w equal to z how did we get g of w equal to z first of all for any w such that mod w-w0 is less than delta there is a z such that w equal to f of z that is because of the fact that N of w equal to 1.

So, there is a certain z take that z okay then f of z that z will be w alright now my claim is that f dash of z is non-zero see if f dash of z is 0 that means you are saying that f the derivative of f-w is 0 that means the 0 z of f-w has order greater than 1 okay see f dash of z for z in mod z-z0 less than row and with w equal to f of z has to be non-zero. Why because if it is 0 if f dash of z is 0.

Then the derivative of f of the it will contradict of N of w equal to 1 you will get N of w greater than 1 for otherwise okay which is not correct N of w strictly equal to 1. What is N of w, N of w is the number of times f assumes the w and how should you think of it you should think of it as the number of zeroes **f** f of zeta-w N of w is number of zeroes of f of zeta-w. And z is as 0 of f of zeta-w because z is f of w sorry w is f of z.

So, z is a 0 of f of zeta-w and it is a 0 of order 1 because if it 0 of order greater than 1 which is what will happen if the derivative of f of zeta-w with respect to zeta is 0 which is a same as saying at f dash of z is 0. Then it will then this N of w will become 2 at least 2 but N of w is 1 so, that fact forces that you know f dash is non-zero okay. So, what I want to tell you is that is you also get that f dash of z is no-zero for all z in f inverse of that disc.

So, when you take the inverse image of this disc here you will get some you will get a open neighbourhood of z0 okay. And the fact is that open neighbourhood will be an neighbourhood restricted to which f is not only one to one the derivative of f will also not vanish. And it is the non-vanishing of derivative there which actually makes this into analytic function. Because what is because how do you get the g of w is actually z that unique z for which f of z equal to w how do you get that.

You get it by a applying residue theorem here see **1/** 1/2piei integral over mod zeta-z0 equal to row of this function zeta f dash of zeta / f of zeta-w d zeta this is actually residue of zeta f dash of zeta/f zeta-w at z at z this is residue. Why because see the for this function, this function has a simple pole at zeta equal to z which lies inside this inside the region bounded by this contour that is the only simple pole.

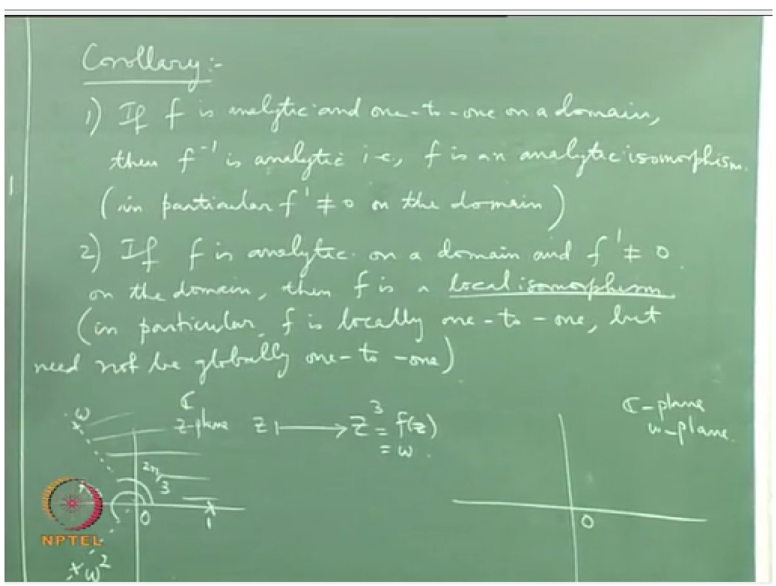
So, the fact is that when you take the residue of this function at z okay because this is a residue at a simple pole so, here this is I am just applying the residue theorem. So, when you take residue of simple pole this is just limit zeta attends to z of zeta-z times that function zeta f dash of zeta/f zeta-w/but what is w, w is fz okay. then if I put push this zeta-z to the denominator I will get limit zeta tends to z fzeta-fz /zeta-z which is f dash of z.

So, I will get f dash of z in the denominator and the numerator will become z f dash of z and I will have to cancel of this f dash of z in the numerator and denominator. And that will give me z this is just by the residue theorem but to show that so, what does it show it shows that g of w is that z is equal to this z. So, in other words it tells you that w is g inverse z. But w is f of z so, it tell you that g inverse is f okay.

So, this is just residue theorem and to apply this I will have to cancel of this f dash of z when I take the limit here. And when can I cancel f dash of z I can cancel f dash of z only if f dash of z is non-zero the derivative does not vanish. And why is that guarantee that is again guaranteed because of N of w equal to 1 that is what I want you to understand this is the crucial point because N of w is 1 f dash of z is non-zero alright for the unique z

I z f of z equal to w therefore I can cancel this f dash of z here and get the fact that g of w is actually z which is f inverse w okay. So, that is the technical point so, now if you put all these things together what you actually get is a following.

(Refer Slide Time: 34:01)



So, let me write me this as a theorem or let me write this as a corollary that is to you know this is to make you this to sum up the relation between when a function is one to one how that is related to the fact when a function has non-zero derivative. So, what are the two situations so, the first statement is if f is analytic and one to one on a domain.

Then f inverse is holomorphic or analytic that is f is a holomorphic isomorphism or analytic isomorphism. So, see the first corollary is that one to one analytic function is a isomorphism one to one holomorphic function is a holomorphic isomorphism it is a very very strong statement corresponding statement for real value function is not true. A one to one real valued differentiable function okay is not differentiable isomorphism.

You can find a one to one real valued differentiable function you can find a one to one maybe even infinitely differentiable function okay which is one to one. But for which the inverse function is not differentiable for example simply take the function from the real line to the real line given by x going to x cube just going just take the function x going to x cube. The inverse function is well defined by the cube root okay which exist that real cube root alright.

So, it is a one to one function and it is infinitely differentiable x cube is just a polynomial so, it is infinitely differentiable. But if you take the inverse function which is x to the 1/3 is all differentiable function at the origin okay. So, the moral of the story is a one to one differentiable function of one real variable is not the inverse function need to be differentiable. But that does not happens in functions of one complex variable as you know whatever happens in functions of one complex variable is completely different from what is what happens with functions of one real variable.

And this is one remarkable thing that happens that anyone to one analytic or holomorphic function is automatically an isomorphism. The inverse function also becomes holomorphic it becomes analytic that is a very deep fact okay. And how do you prove this statement, how do you prove the statement like this I am saying this proof follows from here from our argument see with us.

You see the moment you say f is one to one okay the moment you say f is one to one what did means is that if you fix a point z0 in the domain where f is one to one okay. The fact is that f is one to one will tell you N of w0 will be 1 the fact the saying that the function is one to one means that if you fix a point z0 in the source domain and you take it is value f of z0 as w0.

Then it tells you that z0 is a 0 of all simple 0 it is a 0 of order 1 of f of z-w0 so, N number of times f assumes the value w0 is 1. But then by this argument the number of times the function assumes the value any nearby values will also be 1. And I told you that this argument actually tells you that the derivatives cannot vanish. Because the movement the derivative vanish is then the number of times that the function takes the value at that point goes up okay.

So, the moral of the story is that f is one to one then you are in the situation this N of w is will become 1 and this N of w becoming 1 will force that f dash is non-zero. So, in particular if f is analytic an one to one on a domain then the derivative of f cannot vanish derivative of f cannot vanish and f inverse will be defined f inverse will be defined and it is holomorohic by this argument.

And mind you to check a function is holomorphic it is enough to check at every point so, the argument that we have here I can go through the same argument. I can fix a z0 then I can this w0 then I will get that f inverse will be holomorphic. When I restricted to this disc but then I can cover the target domain by all by such discs by changing z0 in the source domain. So, I will get that the inverse function f inverse is holomorphic at every point.

I mean after all holomorphic at one at a point means it has to be holomorphic in a or analytic in a disc surrounding that point. And I can cover the whole region. So, I can check that f inverse is holomorphic locally but that is could enough to check that f inverse is holomorphic. Because checking holomorphic holomorphicity or analyticity is a local check okay. It is like checking continuity if you want a check a function is continuous one as on a z.

It is enough to check on an open cover okay similarly if you want to check a function is holomorphic or analytic it is enough to check on a open cover. So, I can take the open cover to be given by such discs okay. So, the moral of the story is if f is analytic an one to one on a domain then f inverse is analytic. And f is an analytic isomorphism in particular f dash is not 0 on the domain f dash can never vanish okay.

So, if it is one to one analytic it is very strong condition it is terribly strong condition. It tells you that it is a isomorphism okay and in particular it also tells you the derivative will never vanish.

This is a first corollary what is the second corollary the second corollary is the other condition instead of assuming f is 1 not one to one suppose I using derivative of f does not vanish.

Then what do I get okay the answer to that is the second corollary which says that case you will get a local holomorphism okay. So, let me write that if f is analytic on a domain analytic cor holomorphic on a domain and f dash if not equal to 0 on the domain by which I mean that f dash never vanishes the derivative. The first derivative of f never vanishes on the domain then f is a local isomorphism it is a local it is not a in particular f is locally one to one.

But necessarily but need not be globally one to one okay, so you know so, the whole purpose of this lecture was to try to tell you what is the difference between assuming one to one on the function f which is analytic and assuming the derivative does not vanish. If you assumes it is one to one then it is an isomorphism. So, it is very strong condition if you assume it is derivative and if you assume 1 to 1 then the derivative cannot vanish.

But if you assume if you do not assume it is 1 to 1 but if you assume the derivative does not vanish then is need not be 1 to 1 it will be it could be many to 1. But the point is locally it will be 1 to1, so locally it will be an isomorphism by the first part, it will be a local isomorphism okay. so, and what you must understand is that is what it is says, that is what this argument also says what we have said is that if f dash of z0 does not vanish okay.

Then we have found this disc where f we have in the inverse image this disc you know that f dash does not vanish and there it is a an isomorphism already proved and this happens for every point z0 locally okay globally it need not be an isomorphism. And what is the illustration for this is the I will give you a simple illustration for this, that will help you to think take the function z going to z cube.

Simply take the function z going to z cube, so you know here is the source complex plane, z cube is f of z which is my w okay, so this is the z plane which is the complex plane and then here is the target plane which is the complex plane it is the w plane and you know what will happen is 0 will of course go to 0 original go the origin but if you take a point different from the origin.

Then funny things will happen in fact as you will know you know I take this ray which for which this angle is 2 pie by 3 okay is the ray on which the and then I take this other ray which is mirror image of this which is for which the angle is 4 pie by 3 that is 240 degrees is 120 degrees okay. This is 4 pie by 3 and you know these the cube roots of unity will lie on these 3 the so 1 will lie here.

And then this is omega complex cube root of unity and then there will be another the other one is of course as you know omega square they will lie on this ray and the fact is what this function will do is it will map a whole sector, this whole sector of 120 degrees will be mapped on to the whole plane. Because you know any point here can be written as R e power I theta z=R e power I theta if I apply z cube I will R cube e power i 3 theta.

So, as theta very soon 0 to 120 degrees, 3 theta will go from 0 to 360, so this whole sector will be mapped on to the whole disc I mean I wanted to the whole plane. Similarly this sector from 120 degrees to 240 degrees that will again be map once more to the plane and from 240 degrees 236 degrees that will again we mapped once more, so this so as you go around once if you take the unit circle okay which passes and you go around once from 1 to omega to omega square and come back the unit circle there will go 3 times.

The unit circle will be traverse 3 times and what is going to happen is you see if you omit the point 0 here then you can omit the point 0 there. And the about the derivative, the derivative is what, derivative is 3 z square, the derivative will not vanish if you omit 0. But, so the derivative is non 0 but still the function is not 1 to 1 is 1 to 1 only on this any single sector if you take there it is 1 to 1 on the whole disc 3 to 1 okay on the whole it is 3 to 1.

So, what it will what so what you get is it is many to 1 but if you restrict it to each of these sectors it becomes 1 to 1, so it becomes locally 1 to 1 if you omit the origin. And what is this what is each sector it is a inverse image of the deleted plane the punctured plane. So, you take the target complex plane remove 0 and take the inverse image then each of these will be each sector here with the origin removed there will be 3 inverse images.

And on restricted to each of this will be a holomorphic isomorphism okay and the problem is at the point 0 because the derivative vanishes and at that point there is no neighbourhood of that point where the function will be 1 to 1 and that is the reason why the points at which the derivative of an analytic function vanishes they are called critical points because you cannot find any neighbourhood where the function will be 1 to 1.

It will however smaller neighbourhood of that point you take the function will become many to 1 okay whereas the derivative does not vanish you can always find any neighbourhood is 1 to 1 in in fact even a holomorphic isomorphism that is what the second corollary says okay. So, this is an illustration of both of these things okay, so I will stop here.