Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-08

Completion of the Proof of the Inverse Function Theorem_ The Integral Inversion Formula for the Inverse Function

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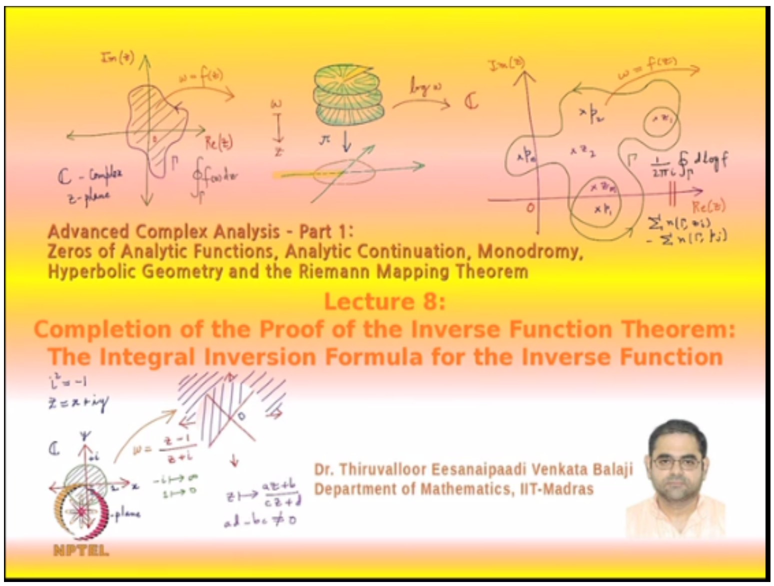

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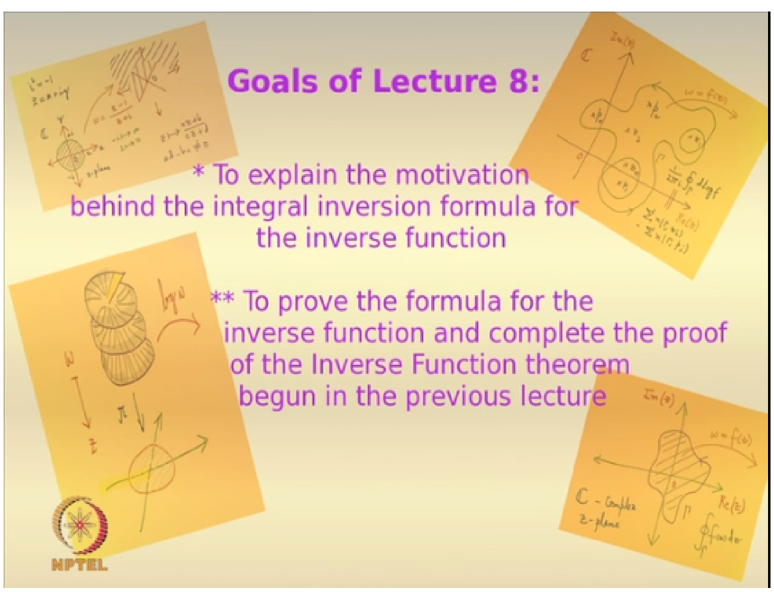

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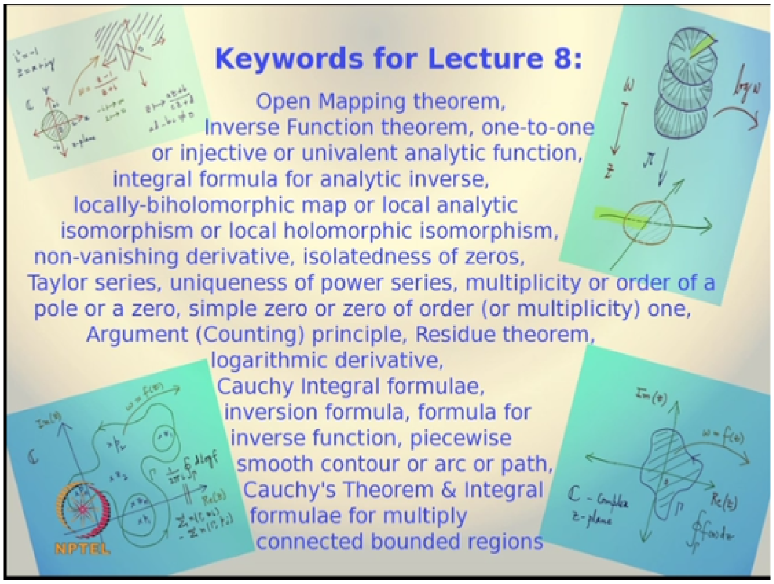

Okay, so **so** let us continue with our discussion about the inverse function theorem.

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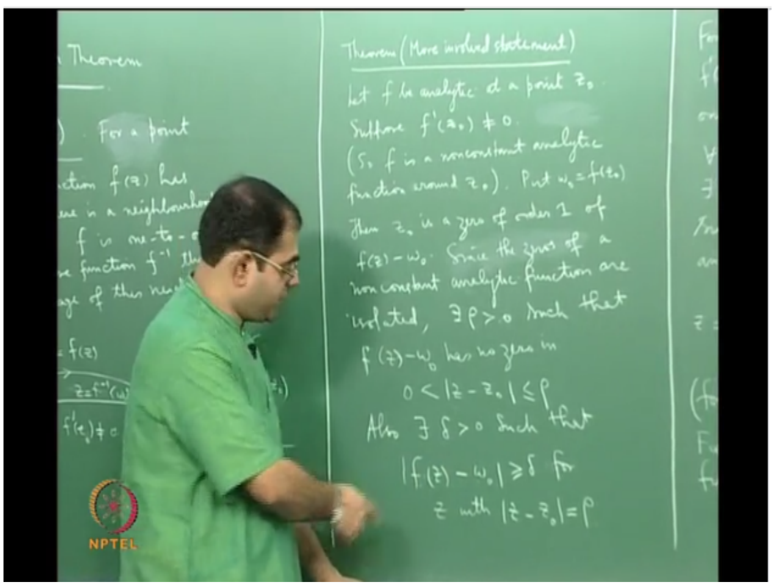

So, let me quickly recall you start with the analytic function analytic at a point z0 assume is the first derivative at z0 is non 0, that means you are saying that z0 is a 0 of order 1 of the function f of z-w0 which is also a non constant analytic function and then you choose a disc centred at z0 radius row such that on the closed disc z0 is the only 0 of f z-w0 and on the boundary of the disc fz –w0 does not vanish. So, if you take it is modulus that is bounded below by a certain delta and for this delta.

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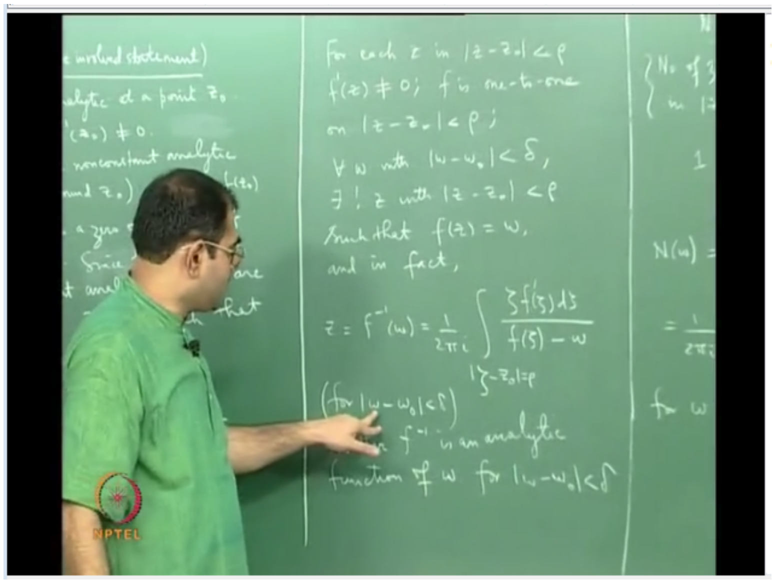

You look at all w such that mod w-w0 is less than delta. And the claim is that the that f is actually one to one on this disc. And the inverse of f is given by this formula okay so, what will what we have prove so, for is that we have proved that f is one to one on that disc.

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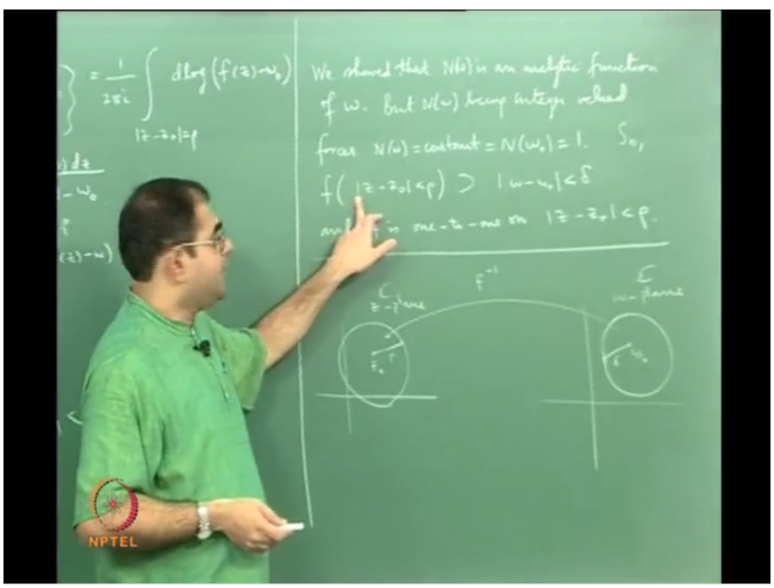

We have prove that f assumes every value w such that mod w-w0 is less than delta every such value is assumed by f in this disc centred at z0 radius row okay. And which is restatement of the fact that that f is one to one on this disc okay I mean given the fact that the number of times it assumes the value f assumes the value w is exactly the same as the number of times f assumes the value w0 for any w.

And but the number of times it assumes the value w0 is 1. Therefore the number of times it assumes the value w is also 1 okay. So, you get both the you get the one to one nature a nature of f okay. Now the only thing that I have to show is that this is the this formula for every inverse w is correct okay. So, let me try to explain at least the logic behind that formula okay.

So, you know our so, let us go back to this so let us back to this diagram see you have on the so, you have on the z plane. We have this point z0 and we have this disc centred at z0 radius row okay. And then on the complex plane that is given by the omega w plane again we have this, this other disc centred at w0 radius delta. We have this disc okay and of course f of z0 is equal to w0 okay.

And f inverse is kind of you know defined from here to here this is f inverse okay it is at least defined properly as a function. Because f is one to one restricted to disc to this disc alright. And think for a moment as to why this formula why should you be able to you know expect such a formula first of all you should get a feeling for why you should get a formula like that okay.

So, what I am trying to say is see do the following thing choose I mean it is I just want to say that this formula is a it is just quotients integral formula for f inverse. If you look at it very carefully so, you see what is so, let me you know go back and let me rub this of so, let I can use this side of the board and let us for a movement you know recall the quotient integral formula. The you know the first order quotient integral formula or the zeroth order quotient integral formula okay the first of the integral formulas.

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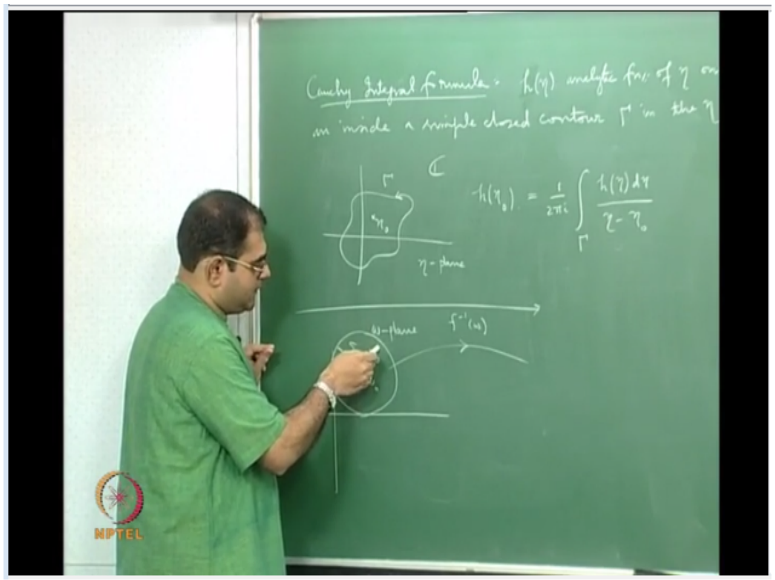

You see suppose you have so, use the quotient integral formula mod is a integral formula so, you know you assume that f let me have not use it let have f let use h okay and let me use some other variable. So, let me use instead of using z let me use neta okay h neta analytic or holomorphic analytic function of eta on and inside a simple closed contour gamma in the neta plane.

So, you see the diagram is something like this I have the complex plane and I have the variable is neta so, I am calling at the neta plane okay. And by bus using neta because I will not use z okay and there is some well there is some simple closed contour gamma okay. And mind you so, it is a closed curve it is simple that means it has no self intersections. And it is a contour means that it is piece wise smooth.

That is it is a continuous image of an interval that is what it is say that it is a path or a continuous path further you want it to be a piece wise smooth which means that if you parameterise that then as a function of the parameter it should be not only differentiable but the derivative with respect to the parameter should be continuous and that should be at least piece wise continuous on the interval okay that is what it means.

So, mind you all this is needed to be able to do the path integral on a contour okay. So, then if I take a point neta0 here okay what is quotient integral formula. It says that if you calculate 1/2piei integral over , of h of eta d eta by eta-eta0 what will you get. You will simply get h of eta0 okay this is the quotient integral formula. This is the first quotient integral formula right of course if you differentiate both sides with respect to eta0.

And you allow that differentiation to go pass the integral sign you will get this successive higher quotient integral formulas okay which will tell you that the higher derivatives at eta0 are expressible in terms of this integral with a factorial appearing on top and then you have higher powers of eta neta-neta0 appearing in the denominator okay alright. You so, let us supply this tower situation in our situation we are on the w plane okay.

And on the w plane there is that point z0 I am sorry there is a point w0. And there is that disc centred at w0 radius delta okay. And we are having this function f inverse of w you are having function f inverse of w. You see finally we are going to our aim is show that f inverse w is an analytic function okay of w that is our aim okay. And then we have to show that the analytic function is given by this formula.

This is what we want to prove but for a movement to get an idea as to how is formula came the first place let us assume cut f inverse w is actually an analytic function of w okay. Then see take the w here okay and what you do is well you know see it is as I explained here the image of this disc contains the disc centred at w0 and radius w okay. In fact it will also contain more or less it will contain the boundary of the disc okay.

And what I want to say is for example you know if I take a small curve here gamma surrounding w okay. And if you believe the that f inverse w is an analytic function of w and what then what will be the quotient integral formula give you for f inverse w. It will tell you that if I take 1/2piei and integrate over gamma f inverse of neta d in eta by neta-w. I am supposed to get f inverse of w.

This is just the quotient integral formula the quotient integral formula says that mind you whenever I write this neta the d neta where the movement I write d neta neta is a dummy variable which lies on gamma okay. So, I did not want to write f inverse w d w because I want to put then will I have to put w-w it does not make sense. I wanted w here so, I have change the dummy variable to neta okay.

And this is just the quotient integral formula which is valid provided f inverse is analytic function of w okay. Now you do the following thing you put zeta to be a f inverse neta you put this if you put zeta equal to f inverse neta what you will get is that means that f of zeta is neta alright. And it will mean that d neta will be a f dash of zet ad zeta so, f dash of zeta d zeta will be d neta okay.

If you write it like this symbolically the element of integration okay and you know this then this formula you know mind you f is of course f is analytic and I have assumed f inverse is also analytic. Therefore f and f inverse are isomorphism mind you okay they are isomorphism's and therefore what I am doing is I am simply in this integral I can make a change of variable.

If I can make a change of variable what will happen to that integral that integral after change of variable will simply become I will get the following things. So, let may write it here you will get f inverse of w is equal to 1/2piei integral so, now you see I am changing the variable from neta to zeta okay. And now since neta lies in the w plane f inverse neta will lie in the z plane.

So, zeta will lie in this z plane okay so, what will happen is that I will have to take f inverse of this curve okay. I am making a change of variable in my integration and if I plug it here I will get so, let me plug it here f inverse neta so, I this will become zeta f dash of zeta d zeta divided by neta is f zeta-w okay. And if you look at it this is as same as the formula here that is the same formula that as appeared here only thing is that the boundary curve.

That I have taken which should be f inverse gamma is actually the is simply d circle centred at z0 radius row with the positive orientation of course okay. And that disc can be replaced by that, that f inverse gamma can be replaced by this boundary curve is justifiable because in the region between the two curves there are going to be no singularities okay.

The version of quotient theorem says that between two curves one containing inside the other if there are no singularities then the integral over the smaller curve is the same as the integral over the bigger curve okay. So, in principle I can actually replaced this f inverses gamma by mod z-z0 equal to row. You must understand that since f inverse is analytic okay since f inverse is one to one f inverse of gamma will also be a simple closed curve.

Because it is a one to one continuous and it is conformer it is analytic. So, this will also be a curve simply closed curve it will be a simple closed contour okay. And the fact is that this can be replaced by that okay so, the moral of the story is that if you assume that the f inverse w is analytic as a function of w. Then it is very clear how this formula comes this formula is not mysterious okay.

You can derive it in this since it is just other avatar of the quotient integral formula for f inverse w okay let removes them mystery out of this formula now what you do is that is just to you know cyclonically satisfy you that this is not something mysterious. This is something that you can guess very easily okay so, there are two things in mathematics one thing is to guess the right formula then the second thing is of course to really prove it without any you know gaps without hand waving.

So, now you have to prove this formula that we have to prove that this is the correct formula that can be done in a completely a different way. Once you believe the formula you can always find so many ways to prove it so, cycalogical difficulties is always in trying to first of all believe that is that formula is true okay. So, that is also very very important so, that is I demonstrated now. So, let us come back to proving this formula. So, maybe I will let me go somewhere for example okay so, let me rub this half where I can just part of the board I do not need this anymore let me continue here okay.

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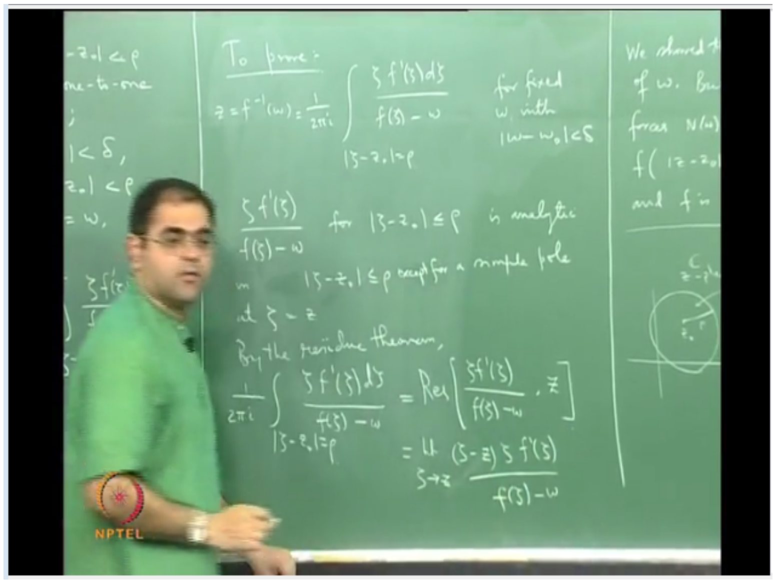

So, will let us try to prove this to prove f inverse of w mod w-w0 less than delta okay this what you have to prove. This is the inversion formula is what you want to prove okay. The proof is pretty easy because you look at the function zeta f dash of zeta by f of zeta-w look at this function for mod zeta-z0 strictly less than or equal to row look at this function on this closed disc okay.

That means I am taking the boundary curve and also the interior on this if you see this is a analytic function except for a simple pole at zeta equal to z see it is analytic in the closed disc except for a simple I should not say with except for a simple pole at this point see that function is a analytic on the closed disc. The only place where it will lose analyticity is when the denominator vanish.

And the denominator well will vanish exactly at zeta equal to z okay. And why that is because zeta equal to at zeta equal to z f of zeta is f of z which is z equal to w. And therefore the denominator 0 that means z is a 0 of a f of zeta-w but 0 of what order it is 0 of order1 because f is one to one it is only the 0 of order1 okay. Therefore it is a 0 of order 1 of the denominator so, it is a simple pole.

So, it is a function that has a simple pole at only one point in the closed disc every other point it is analytic. Now what is a residue theorem the residue theorem says that you take a if you integrate a function over a boundary curve and divide by 2piei okay. Then you will simply get some of residues of f at the poles so, by the residue theorem 1/2piei integral over mod zeta-z0 equal to row of zeta f dash of zeta d zeta by f of zeta-w is actually the residue of the function zeta f dash of zeta by f of zeta-w at the point z.

Because z is the only pole and it is a simple pole and what is the residue at the simple pole it is by taking the limit as z as the variable tense the pole of the function multiplied by the variable-the pole okay. So, if you calculate this, this is actually limit zeta tense to z zeta-z multiplied by this function zeta f dash of zeta by f zeta-w okay. And if you compute this, this is actually equal to z okay.

That will show that the value of this is equal to z in other words the value of this is equal to z which is f inverse w and your formula is proved okay. Why is the value equal to z very simple you can write it out at pretty easy to see this is actually limit zeta tends to z.

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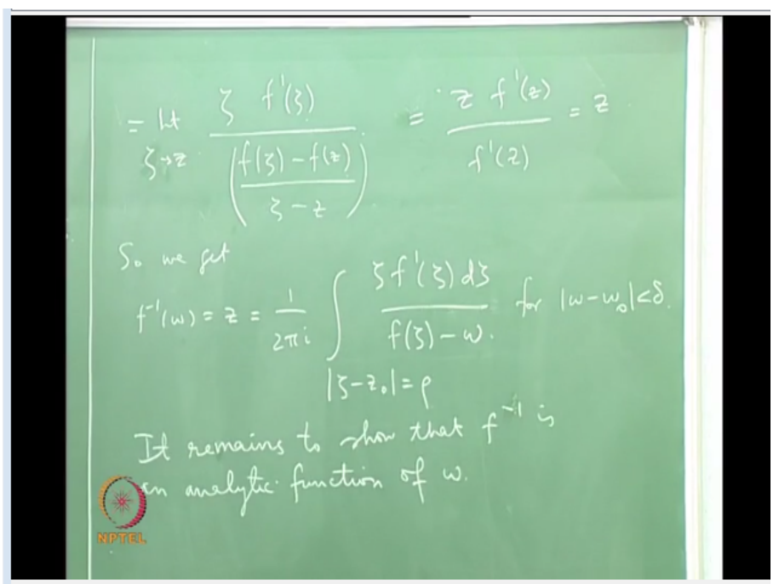

You see I will get zeta f dash of zeta by let me group this f zeta-f of z because w is f of z mind you divided by zeta-z zeta tensed to z. But m is zeta tends to z of this is precisely f dash of zeta. It is f dash of yeah limit zeta tends to this is f dash of z okay. And limit zeta tends to z of the numerator is z f dash of z. So, what I will get is this will just be z f dash of z by f dash of z that is simply be z.

And that calculation is valued because f dash of z is not 0 f dash is non-zero okay so, that is something that you will have to convince yourself that is f dash is non-zero should be obvious okay. So, if you cancel f dash of z out you will get z and that proves this formula okay and so, **so** will get f inverse w is z is this 1/2piei integral over mod zeta-z0 equal to row zeta f dash of zeta d zeta by f of zeta-w.

You get this so, you get this formula the inversion formula okay now the only thing that you will have to worry about is the statement that f inverse is an analytic function of w for w lying inside this disc. And so, in other words you have to show f inverse of w is actually differentiable function of w for w lying in that disc centred at w0 and radius delta okay. And in fact the proof that it is differentiable in that disc is you know it is a same kind of technique that we have already seen.

It is a same kind of technique that we proved I mean which we used to prove that n of w that we defined in the proof of the open mapping theorem. The function n of w which is just the number of tomes the function f assumes the value w. You proved n of w is actually an analytic function of w it is a same technique that you have you that have to use to show that f inverse w as defined like this is an analytic function for w analytic function of w for of course mod w- w0 lesser than delta okay.

This is something that can be shown right so, let me outline that so, that is the only thing that is left out to show that f inverse is analytic function of w okay. So, probably I will write the now I can now let me rub of this so, **so** let me write here it remains to show that f inverse is an analytic function of w okay. This is what is this only thing that is we prove mind you that f inverse is already continuous.

You know why that is because you see f is already an open map okay f is non-constant analytic function we have already in an earlier lecture that an non-constant analytic function is an open mapping. So, f is an open map but when f is an open map it means that if f has an inverse that f inverse is continuous. So, the continuity of f inverse is already there so you must remember that because of the open mapping theorem f inverse is continuous already that is already there okay.

Now we will now we can prove the analyticity of f inverse with respect to w okay that can be done as follows. So, let me just indicate what needs to be proved and then we can when you can fill out the details. So, you know let us instead of using instead of calling this function has f inverse w let us give it some other name let me call it as h of w. So, that is easier to write.

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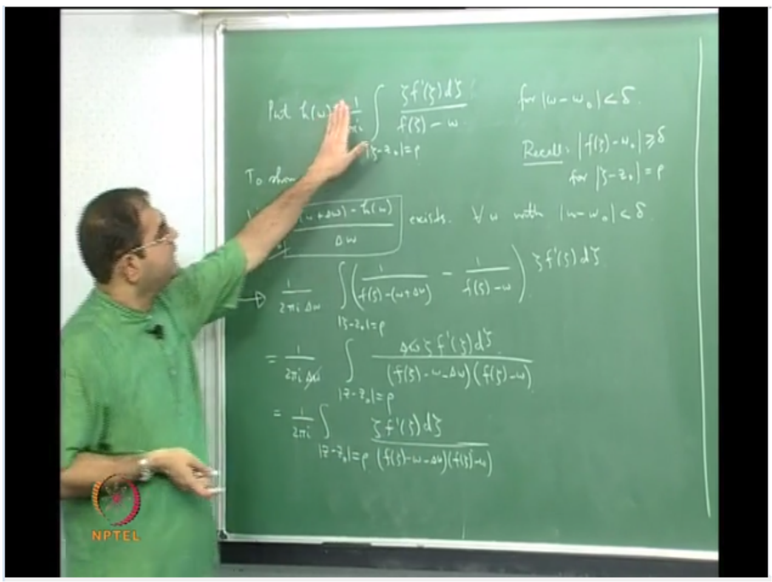

So, put h of w is equal to 1/2piei integral mod z-integral zeta- z0 equal to row zeta f dash of zeta d zeta by f zeta-w for mod w-w0 strictly less than zeta put this okay recall that is here mod of f zeta –w0 is always greater than or equal to delta for zeta-z0 equal to row I mean this is how we chose the delta first. And then and I mean you first chose the row and then it the delta after that.

And this how we chose okay row and delta now what I have to show I will have to show that h is analytic function of w okay. So, what I will do is I will have to calculate h of I will have to calculate I will have to show limit w delta w tensed to 0 h of w+delta w-h of w by delt aw exist. I have to show that this limit exist for every w with mod w-w0 lesser than delta this, what I have to show.

So, I show this I am showing that h is differentiable on the disc mod w-w0 less than delta and that the fact that the functions are differentiable at every point in a disc means it is analytic in the disc. Because analytic is basically differentiable once at every point in an neighbourhood of a given point right. But what is this quantity this is this quantity okay so, for those of you for watching this lecture and exercise is to just stop at this point.

And then just recall how we prove the function n of w is an analytic in the case of the proof if the open mapping theorem and literally follow the same ideas okay. But anyway let me continue so, what is this quantity this quantity is if we calculate it is in so, it is 1/2piei and then there is a delta w and I will get integral over mod zeta-z0 equal to row. And I will get zeta f dash of zeta.

So, there is a zeta f dash zeta is a common fact it is a common term so, what I will get is the I will get simply get 1/f of zeta-w+delta w-1/f zeta-w this whole thing into eta f dash of zeta d zeta. This is what this difference will be okay and if you compute it, it will be 1/2piei delta w well if I take L.C.M then I am going to end up with integral mod z-z0 is equal to row I am going to get a delta w on top zeta f ash of zeta d zeta divided by f zeta-w –delta w into f zeta-w okay.

And then since delta w as got nothing to do with the variable of integration zeta okay so, I can cancel this delta w out. And what I get is a I will get 1/2piei integral over mod z-z0 equal to row of zeta f dash of zeta d zeta over product of these two terms f zeta-w-delta w into f zeta-w okay. And what I am suppose to how is that the limit of this quantity as the delta w tensed to 0 exist okay.

You one is a very nice one can one will pass the let the limit pass through the integral and if you apply the limit if you allow the limit to go inside the integral okay which is not always possible. But if you nicely do it what will happen is that the limit will be 1/2piei integral over mod z-z0 equal to row zeta f dash of zeta d zeta over f zeta-w the whole square that is what you will get which is the expression that you will get.

If you take this and differentiated with respect to w if you what h dash of w is differentiation of h of w with respect to w. So, you have to differentiate this with respect to w oaky that means you are double it d/dw to this expression. But then you allow nicely the d/w to pass through the integral if you allow that then you will be only differentiating the what is the integrant and the variable with which you are differentiating is w which is difference from the variable of integration okay.

So, if you differentiate this integrant with respect to w you will simply get zeta f dash of zeta by f of zeta-w the whole square that is what I going to get. And that is what you are getting here if you let the limit delta w tend to 0 pass into the integral okay. So, the moral of a story is from here how do you prove that the limit of this as delta w tends to 0 is this expression with f zeta-w the whole square the denominator oaky. So, let me write that out uh so, let me get rid on this.

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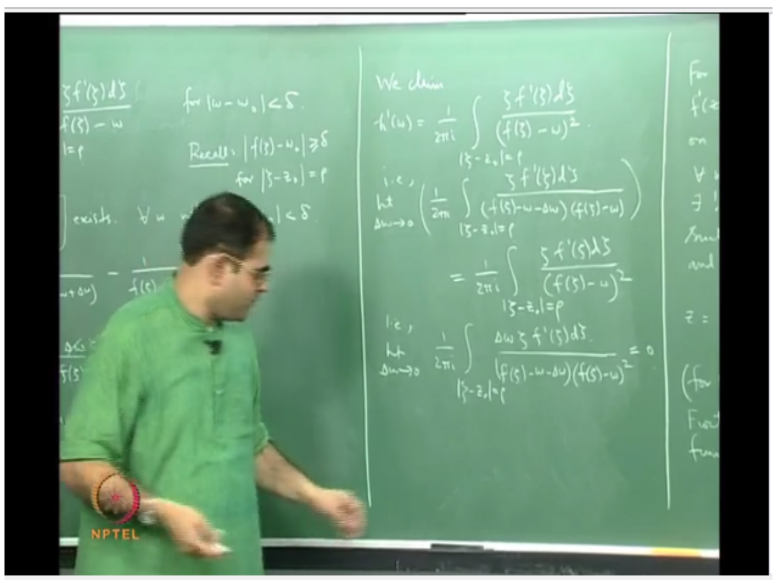

So, I can use that part of the board see we claim h dash of w is actually 1/2piei integral over mod zeta-z0 equal to row into differentiate the integrant with respect to w which is zeta f dash of zeta d zeta over f zeta-w the whole square. We claim this is this okay so, in other words you are claiming the limit as delta w tends to 0 of this quantity is that, that is limit delta w tends to 0 of 1/2piei integral over mod zeta-z0 equal to row of this quantity which is zeta f dash of zeta d zeta over f zeta- w- deltaw times f zeta-w is equal to this quantity which is 1/2piei integral over mod zeta-z0 equal to row of zeta f dash of zeta d zeta over f of zeta- w the whole square.

This what you are claiming so, you have to show limit of this quantity equal to this which is effectively differentiating under the integral sign. And it tell you that, that is always possible in Cauchy theory I mean that is whole point about studying analytic functions okay. In fact I told you that quotient integral formula itself is the example of differentiating under the integral sign.

The higher quotient integral formulas are gotten from the first Cauchy integral formula by literally differentiating under the integral sign. And all this as possible because you are working the analytic functions that is the whole philosophy so, you see I have to show limit of this quantity equal to this okay. But now how do you prove that the trick is you absorb this also to the left side and show that the limit of the difference is 0 okay.

And why I can absorb this into the limit is because this as got nothing do with the limit. The limit is with respect to delta w going to 0 and this is got no delta w term. So, that is you have to show limit delta w tends to 0 of if I absorb it on to the left side I will get 1/2piei integral over mod zeta-z0 equal to row of if I take this-this okay. If I take L.C.M what I will end up with this I think I will get it delta w into zeta into f dash of zeta into d zeta divided by f zeta-w-deltaw times f zeta-w the whole square is 0.

This is what it will be it will turn out to be okay because if you push this whole term inside the limit on the left side okay. And then take the integral out and if you take zeta f dash zeta d zeta out and you take L.C.M this is what you will get okay. So, and which is again correct because if you again nively let limit delta w tend to 0 inside the integral okay. There is a delta w on top so, the integrand will vanish.

Therefore the integral is 0 integral of a 0 integrand is 0 if you want to think of it like that so, this is again still you know in still about trying to pass the limit into the integral that is what you want appreciate but we are simplify it to very nice level. But so, but now what you do is to show that the limit of something is 0 you show you use now go back to basic analysis and say when is the limit of certain quantity equal to 0 as delta w tends to 0.

You use the epsilon delta condition you say that given an epsilon greater than 0 you can make the modulus of this quantity smaller than epsilon provided choose delta sufficiently small such that mod delta w can be made less than delta so, provided you can find such a delta okay. So, you go to the epsilon delta definition of a level and that needs you to estimate this integral okay.

And to estimate the integral you use the ml formula which is the formula which says that modulus of the integral is less than or equal to integral of the modulus. So, if you apply that well let me quickly write that out it is pretty easy so, what you do is you to the following thing.

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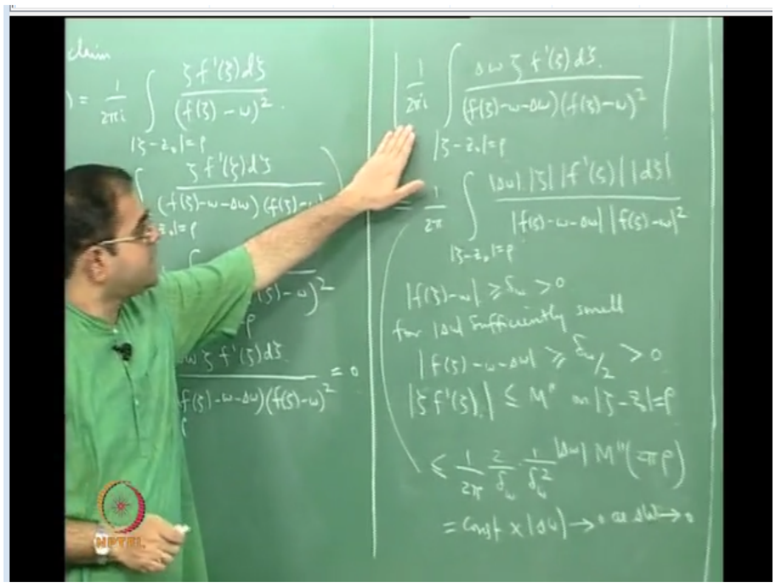

You just estimate that estimate this modulus of 1/2piei integral over mod zeta-z0 equal to row of this guy deltaw zeta f dash of zeta d zeta by f zeta-w-deltaw into f zeta-w the whole square modulus of the integral is lesser than or equal to integral of the modulus so, this 1/2pie integral over mod zeta-z0 equal to row okay mod delta w and I will get mod zeta mod f dash of zeta mod d zeta by mod f of zeta-w-deltaw into mod f zeta-w the whole square okay.

I will get this and the point you have to understand is that finally the if you look at see mod of f z mod of f zeta-w is bounded below by a certain deltaw okay which is positive because you are on this boundary circle alright. And if you recall this is how we proceeded in the proof of the open mapping theorem and for mod deltaw sufficiently small since mod of f zeta-w-deltaw tensed to mod of f zeta-w as deltaw tensed to 0.

You can ensure that modulus of f zeta-w-deltaw is also greater than or equal to deltaw by 2 okay. Because if this is approaching this and this is greater than or equal to deltaw so, you can certainly make deltaw small enough I mean this is small little deltaw okay as this deltaw approach is 0 okay that this is greater than or equal to small deltaw and that this approach is that will tell you that this also can be made lesser greater than or equal to half of that okay which is positive okay.

And that of course mod zeta mod f dash of zeta they are both bounded mod zeta f dash of zeta is lesser than or equal to some mw prime on mod zeta-z0 equal to row. Because z f dash of z is an analytic function f is analytic so, f dash is analytic and z into f dash of z is also analytic and you are look so, it is continuous so, mod of that is continuous and you are looking at a continuous function on a compact set compact connected set.

So, it has a minimum it has a maximum and this is the maximum value so, it is a boundary okay. So, what you will get is that so, all this will tell you and if you integrate over mod zeta-z0 equal to row mod d zeta you will simply get the length of the arc namely the perimeter of the circle okay which is 2pie row so, what will you get is you will get that this quantity I slesser than or equal to 1/2pie into I will get a this is greater than or equal to deltaw.

So, reciprocal of this is less than or equal to 1/deltaw and square of that will be 1/deltaw square so, this will and this will contribute to a 2piedeltaw. So, I will get 2/deltaw then I will get a 1/deltaw square okay and then I will get of course I will get a mod deltaw on the numerator that will come out as it is and this mod zeta f dash of zeta will give rise to an m double prime and what is left out is integral of mod d zeta over this curve which is a 2pie row okay.

So, which is actually constant times mod deltaw this is what it is and that will go to 0 as delta w tends to 0 that will go to 0 as delta w tends to 0. Because this constant is independent of the triangle or deltaw okay so, the moral of the story is that the limit of this quantity as deltaw as triangular deltaw goes to 0 is certainly 0. And that is what you wanted to prove and that completes the proof of that fact that this function is an analytic function of w.

And that function is precisely f inverse w for you recall therefore this tells you that f inverse is an analytic function of w okay and that completes the proof. So, I will stop here.