Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-07
Introduction to the Inverse Function Theorem

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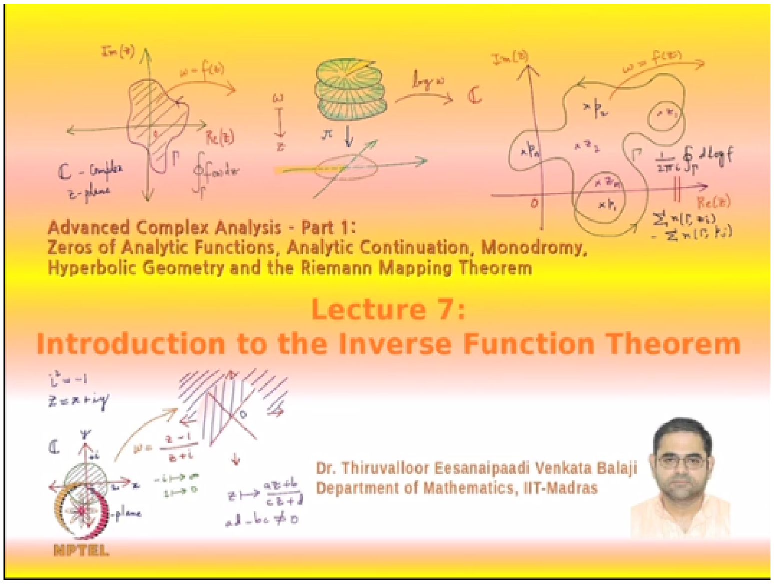

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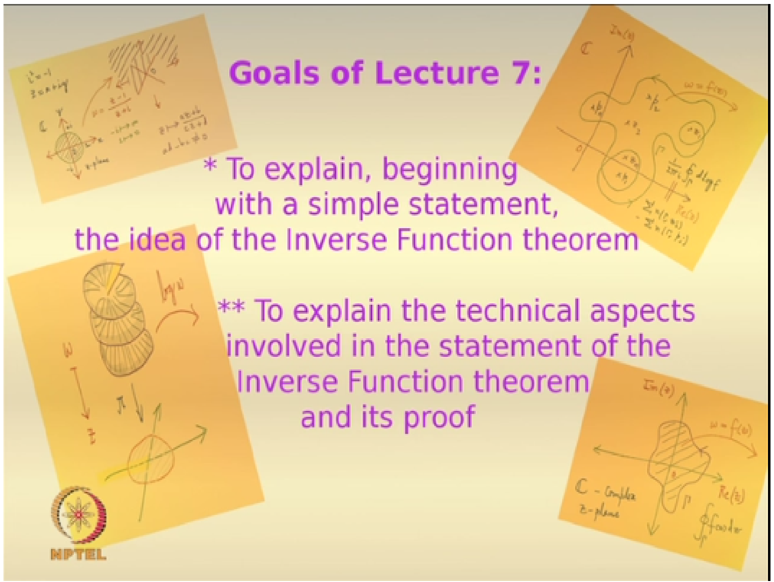

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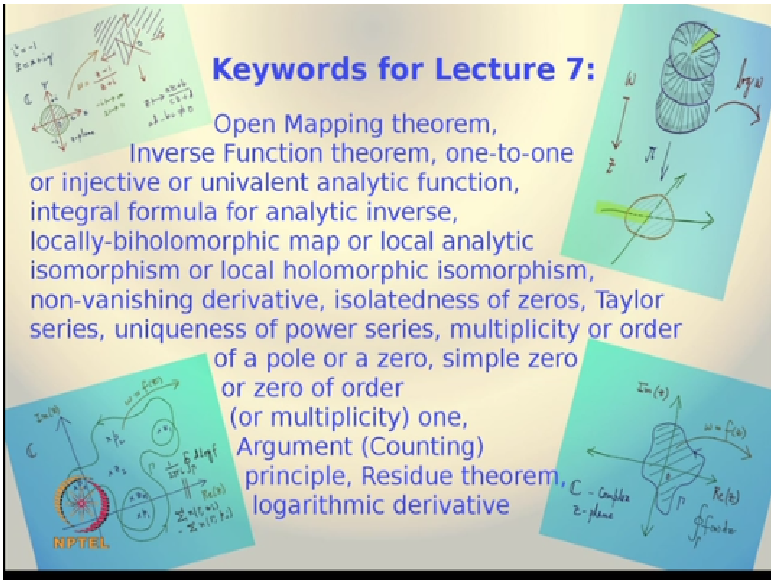

Okay, so let us continue with our discussion so you know the general theme is a theme of zeros of analytic functions and what we saw last time was the so called open mapping theorem which said that if you take a non constant analytic function define on a domain then it is an open map. So, it is maps open sets to open sets the image of an open set under such a map is again an open set okay .

So, what I am going to do now is prove the so called you know the so called inverse function theorem which is essentially the statement that you know if you have an analytic that does not that whose derivative does not vanish at a given point. Then thus the small disc around that point there is a small neighbourhood around that point where the analytic function is 1 to 1 okay.

And not only that the inverse function which can be defined because it is 1 to 1 is also analytic okay and the inverse function is given by an integral formula okay, so this is what we are going to see today. So, let me put that as a title of today’s talk or rather the aim of today’s talk.

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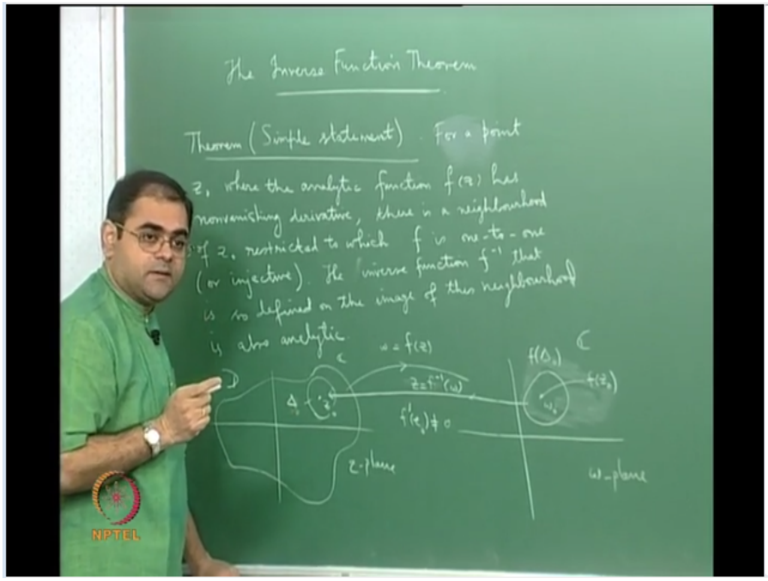

The inverse function theorem, so what is the theorem and well I will call this is simple statement. So, the simple statement is let in every neighbourhood in at every for every point for a point z0 where the analytic function f of z has non vanishing derivative okay which is the statement that f dash of z0 is non 0 okay .

There is a neighbourhood of z0 restricted to which f is 1 to 1 or injective okay. and the function the inverse function f inverse that is so defined on the image of this neighbourhood is also analytic okay . So, this is the statement of the inverse function theorem the simple statement okay, so if I draw a diagram is something like this.

So, here is a so there is some domain I am drawing a bounded domain but bounded domain enclosed by a simple closed curve. But it not look like that it is some domain and by a domain we always mean an open connected set okay. And this is in the source complex plane was is the z plane. And you see on this domain D there is define an analytic function w equal to f of z which takes values in the w plane or the omega plane which is the another copy of complex numbers.

So, this is the w plane and I am considering a point suppose I consider a point z0 such that the corresponding point w0 which is equal to f of z0 it is it image okay. And suppose I assume that f dash of z0 is not equal to 0 so, that means that this is the fact that of put here in the hypothesis that it has non-vanishing derivative okay f dash of z is the derivative of f of z and it is value at z0 so, when I say non the non-vanishing derivative is at z0 okay.

Then the claim is that I can find a neighbourhood for example an small disc surrounding z0 okay where restricted to it is the function becomes 1 to 1 okay. An duh if I call this disc s something let me call this as D0 okay or let me even call t delta0. Then what happens is that f of delta0 will be a neighbourhood of w0. In fact it will be a disc you can vey well expected to be a disc like neighbourhood of w0.

So, **so** let me so you know okay so, I need to remove this and draw something like this here w0 is so, this is f of delta0. If this disc, disc like neighbourhood and this is a w0 is just f of z0. And what happens is that from this image disc from this image f of delta delta0 you have a an inverse function which is given by f inverse. It is given by w z equal to f inverse of w that is inverse function okay.

And the fact is that the fact is first of all that f restricted to this disc delat0 is 1 to 1 okay. There is a disc delta0 a neighbourhood of z0 restricted to each f is 1 to 1 and therefore and the image of this delta0 will be this set f of delta0. You must remember that if you recall the open mapping theorem because f is a of course mind you it has not the derivative at z0 does not vanish.

Therefore f cannot be a constant function so, it is non-constant analytic function because the function is a constant analytic function. Because if the function is constant analytic function then the derivative will be 0 identically 0 so, the fact that the function has non-vanishing derivative at one point tells you that the function is a non-constant analytic function. And what we saw in the previous lecture was that such a non-constant analytic function is open mapping that is it always takes open sets to open sets.

So, if you take this disc delta0 which is this open disc centred at z0 of suitable radius okay. Then it is image will always be an open set okay which I am for convenience like a disc okay. And the theorem they inverse function theorem says that f restricted to delta0 is injective it is one to one therefore you can f inverse is defined for every point w in f of delta0. So, there is a inverse function like this set theoretic inverse function.

That is because for every z there is a unique w for every z you have w equal to f of z and for every w here there is a unique z here such that f of z equal to that w oaky. So, you have a inverse set theoretic inverse map and what the inverse function theorem says that this is not this inverse function is actually analytic okay. And in other words what you are saying is f restricted to delta0 is a holomorphic or analytic isomorphism of delta0 with f of delta0.

It is a holomorphic isomorphism analytic as morphism because this way the map is holomorphic or analytic and f inverse is also holomorphic an analytic. And both are inverse of each other because f is 1 to 1 okay so, this is the inverse function theorem okay. This is simple set okay so, the more complicated statement is is giving you is this statement that will give you what this delat0 exactly is it will give you what this delta0 is it will tell you how to calculate this it will give you a formula for f inverse okay. That is the more deeper statement so, let me write down the more deeper statement here okay.

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So, theorem this is more this I should say more involved statement. What is a more involved statement so, let f be analytic at a point z0 okay mind you this means that f is an analytic defined in a open neighbourhood containing z0. So, it is analytic at point of course means analytic and neighbourhood of the point that is what analytic means okay differentiable and neighbourhood of a point which is the analytic in a neighbourhood of a point.

Suppose f dash of z0 is not equal to 0 okay suppose the derivative of f dash does not vanish at z0 of course it is analytic therefore that is holomorphic. So, it is it means it is differentiable and you know it is differentiable once and it is differentiable infinitely many a times because I just speciality of analytic functions the first derivative exist and higher derivative exist. The fact is the first derivative at z0 is non-zero okay that assume.

That is the assumption choose rho so, the so, let me rub this and say something. I want to say that since you have a non so, this implies that f is a non-constant analytic function. This is something that I should so, f is a non-constant analytic function on in a neighbourhood around z0 okay. So, it is a non-constant analytic function of course when I say around z0 I mean disc surrounding z0 okay.

And you know the if a function is a non-constant analytic function you know that it is zeroes are isolated okay and a non-constant of analytic function for an non-constant function zeroes are isolated. And mind you if I if you take w0 to be the value of f at z0 then z0 is a 0 of f of z-w0 okay. This is the trick that we have always been using all these during all the previous lectures oaky. So, put w0 equal to f of z0 then z0 is a 0 of order1 of f of z-w0 okay.

So, let may explain this of course f if you take the function f of z-w0 okay and if I plug in z equal to z0 of you will get 0. So, z0 is z equal to z0 is a 0 of this function no doubt okay. And the fact that the derivative of this function at z0 is a same as f dash of z0 because –w0 is only a constant. If I take the derivative of this I simply get a f dash z and if a plug-in z0 I will get f dash z0 this is not 0 since the first derivative is not 0. It means it is a 0 of order 1 okay you see if you want may be I can recollect something from say basic complex analysis for you.

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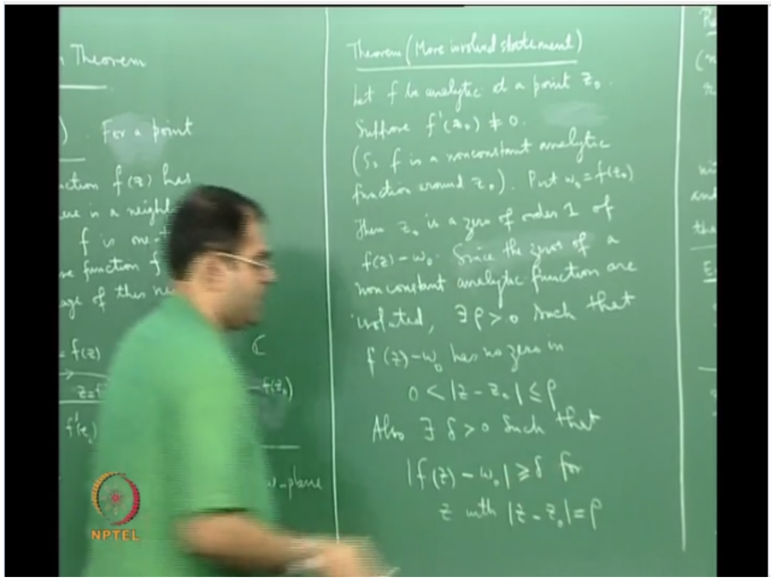

You see you can recall an analytic function non-constant of course non-constant has 0 so, I call the function g of z okay has a 0 of order R at z equal to z0. If you know g of z can be written as z-z0 power r times h of z okay with you know with h of z analytic around z0 at z0. And h dash of h0 is not equal to 0 this is what is meant by a function having s 0 of certain order at the point z-z0.

That means you see function having a 0 of order r z equal to z0 means z-z0 to the power of r can be factored out from the Taylor series of the function centred at the point z0 that is exactly what it is says okay of course if I put z equal to z0 here I will get 0 okay. And the uh order of the 0 is the is this power is the power of z-z0. So, you know what you must understand is that it follows that you know.

If you calculate g dash of z0 and so on and go on up to g you know r-1 of z0 they will always, 0 h should not have a 0 at z0 that is correct we start h dash thanks. Yeah that is see the 0 when you write the function in this form z0 should not be a 0 of this part, that is important that is correct thank you pointing out I miss that. So, you see h of z0 is not h dash you are right h of z0 I snot0 okay.

And see what happens is that if you start differentiating g up to r-1 times and if substitute z0 you will still get 0 okay. But if you differentiated r times you want get 0 okay because if we differentiated r times if you use the product tool and do it okay. Then will you have one term which will involve so, you know so, for example you know if so, you know so, for example if you take g of z let us put z-z0 power let us put 1 into h of z.

Suppose r equal to 1 you know If you differentiate it once you will get you know by product roles z-z0 constant I will get h dash of h+ h of z into differentiation of z-z0 is just 1. And if I calculate g dash of z0 I will simply get h of z0 and that is non-zero. So, if r is equal to 1 then h of z0 is non-zero and g dash of z0 is non-zero okay. And so, you can now, you can generalize it okay .

If you have this situation then g r of z0 will not be a 0 the so, you know g1 g dash is the first derivative g I will put a round bracket here saying that this is the r-1 derivative. But g if I calculate gth derivative at z0 that want be 0 okay. So, this is example when r equal to 1 alright similarly you can try it out for r equal to 2, 3 and so on alright. So, the point is that when you can write g as z-z0 to that power which is the order of the 0 times the function which is does not have a 0 at z0 okay.

And what does this mean see this is just a reflection of the fact that the Taylor series for g centred at z0 starts only from the Taylor quotients are all 0 up to the first r coefficients are all 0 that is what it means see because you know so, you know if you look at it in other way you see what is g of z as a Taylor series. It will be a0+a1z –z0 into +a2 z-z0 squared and so on okay.

And what are the ai the ais are the just the higher derivative of g at z0 by so, an is just the nth derivative of g at z0 by factorial n right. These are the this is the Taylor series alright and if you so, you know the derivatives from the first and of course g of z0 you is 0 mind you. So, a0 is 0 okay a1 is a multiple of the first derivative which I s0 and so on all the first up to the first r-1 a a0 up to ar-1 will always 0.

So, the power series will start with the first term which will correspond is z-z0 to the power of r so, what will happen is that g of z will just be z-z0 power r it will start only a like this. This will be the first I may this will be the first power that you will get and of course the coefficient will be a of course ar. Then you will get then it will start from here it will be ar+1 z-z0 to the power of r+1.

And it will go on like this and the fact is that if you factor out z-z0 power r from this what will you get out the remaining that you will get will be the power series expansion for h of z at centred at z0. And the power series have to be the same because there is an identity theorem for power series. The power series surround expansion at a point the Taylor series which is unique okay.

And it is unique basically because it is given by this derivatives and the derivatives are of course unique oaky. So, what you must understand is this is exactly what has happening when you take a 0 of order are if z0 is a 0 of order r then you know the that is signified by saying that the r-1 derivatives including the function which is thought of a the zeroth derivative okay. They all vanish at that point but the rth derivative will not vanish.

So, in particular you know if you take r equal to 1 that is a situation here in this case z0 is a 0 of fz-w0 okay. And z0 is only a 0 of order 1 because it is the derivative of this fn is f dash of z. And when you evaluated at z0 you will get f dash of z0 and f dash of z0 is non-zero okay. If we had it been as 0 of order greater than 1 then f dash of z0 would have also become 0 okay. So, that should tell you that this is the derivative of I mean it is 0 of order 1 for this function okay.

Now we can find a delta okay so, let me say something else so, let me go back. So, this is about I mean this is all this was just to recall what order means okay. Now go back to this function see f of z is non-constant analytic function f of z-w0 is also non-constant analytic function. And the zeroes of an analytic function are isolated all the zeroes are isolated therefore this is the zeroes of this non-constant analytic function are also isolated.

In particular the 0 z0 is also isolated so, there is a disc surrounding z0 where there are no other zeroes okay. So, since the zeroes of a non-constant analytic function are isolated there exist positive such that f of z-w0 has no 0 in 0 strictly less than mod z-z0 lesser than or equal to rho okay. This is because of isolation of zeroes of a non-constant analytic function okay. So, that in other words I excluded 0 mind you.

Because when I put 0 z has to be equal to z0 and z0 is of course is 0 but this is a deleted closed discs centred t z0 on which there are no other zeroes. And that is such a disc exist because of you know isolation of zeroes of a non-constant analytic function okay. So, there is a rho like this alright now also there exist a delta positive such that the modulus of f of z- w0 is greater than or equal to delta on for z with mod z-z0 equal to rho okay.

I mean this is a fact that we have used couple of times even in the earlier proofs for example in the proof of the open mapping theorem right. So, what will I have to do is I wrote it here so, let me rub this of so, I can continue with the statement of the theorem okay. So, let me continue here so, you see the this is something that we have you know always been using the function f of z-w0 is if we take that function.

That function is not going to vanish on this deleted neighbourhood so, in particular if you take the boundaries of this deleted neighbourhood which is the circle mod z-z0 equal to rho circle standard at z0 radius rho is now going to vanish that circle okay. So, it is a but that circle is both you know it is compact connected and this is mod fz-w0 is a continuous function the continuous real valuehood function on a compact connected set.

Therefore you see image on the real line is going to be a compact connected subset of the real line and therefore it has to b e a closed interval okay. And delta is the minimum value in that closed interval namely it is a left end point to the closed interval oaky. So, such a delta can be found okay the fact is that.

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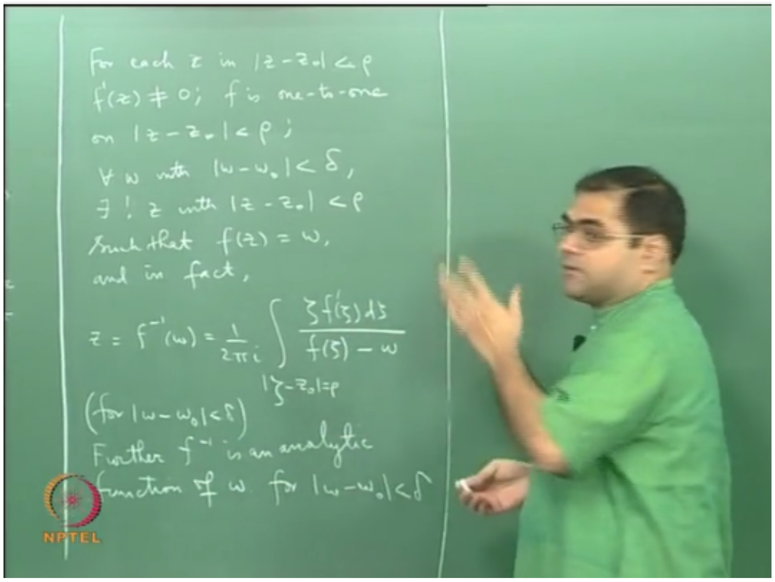

So, what the inverse function theorem says is for each z in mod z-z0 lesser than rho okay f of z is f dash of z is not equal to 0 okay. So, see the derivative you assume is it does not vanish at **at** z0 okay but the fact is that in this disc okay the derivatives never going to vanish at any point that is wants that is one plane okay f is 1 1, 1 to 1 on mod z-z0 strictly less than rho.

This is the disc on which f is 1 to 1 okay for every w with mod w-w0 strictly less than delta okay. This is the target region in the w plane okay there exist a unique z with mod z-z0 strictly less than rho such that f of z is w okay. And in fact z is equal to f inverse w is equal to 1/2piei integral over mod z mod zeta-z0 equal to rho okay h zeta f dash of zeta d zeta by f of zeta-w.

This is the inverse formula for the inverse function so, you see this express is f inverse of w okay of course this is for mod w-w0 lesser than delta okay. So, the formula for the inverse function is given like this and the claim is that f inverse as a function of w is also analytic further of w for modw-w0 less than delta okay. So, this is the strong or the more involve statement of the inverse function theorem.

So, it tells you what is the disc it is a certain disc centred at z0, radius rho okay and it also tells you what is the target disc the target region on which you write out the formula for the inverse that is also a disc centred at w0 and it is radius delta, where delta is the you know minimum value of f of z of mod fz-w0 on for z varying on this boundary disc okay, this is what the inverse function theorem says okay.

So, let us try to prove this, so the first thing I want to say is that you know the first question that I want to think about is the following, so see there are several things I have to prove, I have to show it is 1 to 1 okay. Then I have to show that once it is 1 to 1 than I have to show that the inverse function is given by this formula.

Then I will have to show that f inverse is an analytic function of w okay, these are all the 3 things I have to do, now how do I show it is 1 to 1, I actually go back to the essentially to the proof of the open mapping theorem okay. so, it takes me back to the proof of the open mapping theorem and what is the proof.

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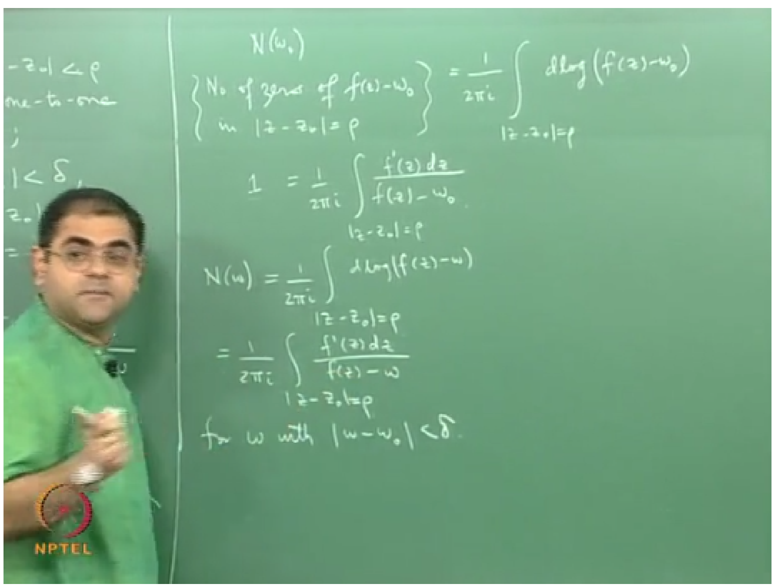

So, you see, so you know let us look at this see what is number of zeros of f of z-w0 in mod z-z0=rho okay. so, it is you know the basically it is the argument principle it is the counting principle which is just counting a number of zeros of an analytic function inside a closed simple closed curve which is just given by the argument principle which is just the residue theorem applied to the logarithmic derivative of the function.

So, what is the number of zeros, if I call that as N of w0 the number of zeros of f of z-w0 in this disc okay this is given by what I mean it is given by well by the residue theorem it has I mean by the argument principle which is residue theorem applied to the logarithmic derivative of this it is just 1 by 2 pie i integral over mod z-z0==rho okay of d log f of z-w0 this is what it is.

If you take d log of a function of an analytic function and integrate it over a boundary curve simple closed curve which is small contour and divide by 2 pie i you will get simply the number of zeros-number of poles okay. But of course is the function has no if it has no poles then you will be just getting the number of zeros counted with multiplicities. And in this case what is this this integral is just 1 by 2 pie i integral over mod z-z0=rho, d log of that is just derivative of that logarithmic derivative which is derivative of this by this function okay what you should understand is when I do whenever you do an integration, the variable of integration lies on the boundary curve okay.

So, for z lying on the boundary curve you know that mod of f of z-w0 does not vanish because what you have assumed is here what you have assumed here is that for z lying on the boundary curve modz-z0=rho, the mod of f of z-w0 is greater than or equal to delta which is the positive number okay and mind you delta is positive because it does not vanish and why this does not vanishes because it does not vanish the only point where it vanishes in this closed disc centred at z0 radius rho is at the centre okay.

And on the boundary it certainly does not vanish, so this quantity it in the denominator that never has a 0 okay. This quantity it does not have a 0 on the boundary, it has a 0 inside and that is what it is counted by this and this is equal to 1 because the number of zeros the number of zeros of this is just the number of times f of z takes the value w0 and that is only once at z=z0, that is our assumption.

Because it has only one 0 and that zero is a border 1 okay, so it is counted only once if 0 is a order of m you have to count it as m zeros okay but this is 0 of order 1, so it is counted only as a one zero and only at 1 point, so you get 1 okay, this is what you get. But now more generally if you recall in the open mapping theorem the proof is the open mapping theorem what we did was we also defined Nw, the number of times the function f of z takes the value w.

And how was that defined that was defined as ½ pie i integral over mod z-z0=rho of d log f of z-w, this is how we defined it and what is this how was this defined I mean which is what is which is equal to the following this is f dash of z dz by f of z-w okay. and for what values of w does this integral make sense this is for w with mod w-w0 strictly less than delta.

You see the fact is the way we have chosen delta is such that that for on the boundary curve the distance of f of z from w0 is greater than or equal to delta okay. So, if you choose a w whose distance from w0 is less than delta then the distance of f of z from that such a w can never be 0. So, this quantity does not vanish on the boundary curve therefore this integral is well defined and this integral will give you the number of times the analytic function f of z assumes the value w in inside this curve inside this circle centred at z0 radius rho okay.

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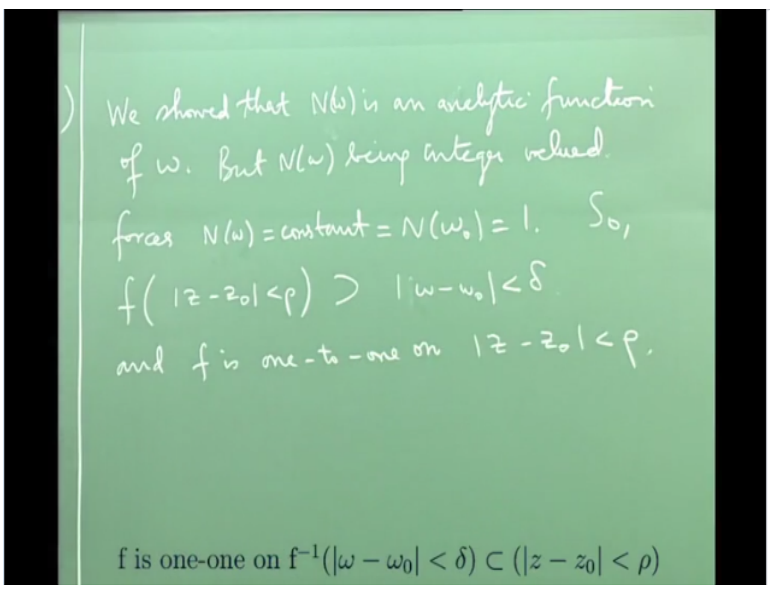

And in fact what we proved last time is if you if you recall we showed in the proof of on the course of proof of the open mapping theorem that N of w is an analytic function of w N of w is an analytic function of w right that is something that we prove and then what we concluded was you see N of w is an analytic function of w it is defined on this disc centred at w0 radius delta.

And but the point is that N of w takes only integer values because it counts the number of zeros okay, it counts the number of times it counts the number of zeros of f of z-w which is the same as counting the number of times f takes the value w for z inside the disc centred at z0 radius rho okay and that is an integer. So, you have an analytic of whose values are in the integers okay such a function has to be a constant.

Because N of w it is an analytic function, so it is continuous an analytic function is always continuous. So, the image of this disc which is a connected set under a continuous map is always a connected set. So, what you should get is that the image of N of w is a connected subset of the integers that has to be single integer. So, what this will tell you but N of w being integer valued forces N of w=constant=N of w0 which is equal to 1.

Because N of w0 is 1 because the number of times f assumes the value w0 is precisely once that is how the disc has been chosen. Because the disc has been chosen to have only one 0 the 0 z0 of f-w0 okay. So, but what this tells you this tells you for every w in such that mod w-w0 is less than delta. The number of times f assume z value w is exactly 1 okay so, what this will you is it will tell you two things.

It will you that the image of the disc mod z-z0 strictly less than rho contains this disc okay every value w which is within a disc distance of delta from w0 is assumed by f at a point z which lies inside the disc mod z-z0 strictly less than rho. So, it tells you two things it tells you that f of mod z-z0 less than rho contains the set mod w-w0 strictly less than delta it tells you this+it also tells you that the fact that it assumes every value once is also telling it is one to one f is one to one.

You had it see the uh what you must understand is the fact that it assumes every value f of z- f of z assumes the value w at a for only one point z okay. And not for more than one point z but that is exactly saying that f is one to one so, we get this is this and f is one to one on this disc okay. So, we have proved this fact we have proved the fact that f is one to one on the whole disc.

And it takes all the values in the image disc it takes each value in the image disc exactly one time okay that proves this right. And so, that is one thing the uh so, that proves if you back to the statement of the theorem. We have proved that f is one to one on this disc you have proved that for every w with mod w-w0 less than delta the disc in the target complex plane w plane okay.

There exist a unique z that there exist a unique z is the 1 to 1 nature of f and that their existent z is because the N of w=1, the fact that N of w=1 means that there is the point z for which f of z takes the value w and there is only one point okay. So, there exist the unique z with z-z0 less than rho such that f of z=w okay, so we have proved that, so what is left out is only to show that the formula for f inverse in terms of w is given by this expression okay.

This is the only thing that is left out, I have to show that this formula holds and I have to show that f inverse is also an analytic function. So, this what is left out, so I will do that in the next lecture okay.